

On p -adic L -functions for $\mathrm{GL}(n) \times \mathrm{GL}(n-1)$ over totally real fields

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Abstract

We refine and extend previous constructions of p -adic L -functions for Rankin-Selberg convolutions on $\mathrm{GL}(n) \times \mathrm{GL}(n-1)$ for regular algebraic representations over totally real fields. We also prove a functional equation for this p -adic L -function, which might be of interest in further study of its arithmetic properties.

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Introduction

In this paper we study the problem of \mathfrak{p} -adic interpolation of the special values of twisted Rankin-Selberg L -functions of irreducible cuspidal automorphic representations π and σ on GL_n and GL_{n-1} for $n \geq 2$ in the sense of [JPSS83, CPS04]. This is in some sense a continuation of the previous works [Bir71, Man72, Maz72, MSD74, Kit94, Sch93, KMS00, Sch01, KS09, Jan11].

We review the general problem of p -adic interpolation and extend the results of [Jan11]. To be more precise, let π and σ be irreducible cuspidal automorphic representations of $\mathrm{GL}_n(\mathbf{A}_k)$ and $\mathrm{GL}_{n-1}(\mathbf{A}_k)$, where \mathbf{A}_k denotes the adèle ring of a totally real number field k . Fix a finite place \mathfrak{p} of k and an integral ideal \mathfrak{N} coprime to \mathfrak{p} .

Assuming that π and σ are cuspidal irreducible regular algebraic, i.e. occur in the cohomology of arithmetic groups, and furthermore that the pair of representations (π, σ) is of finite slope at \mathfrak{p} , we show the existence of a \mathfrak{p} -adic vector-valued distribution $\mu = (\mu^\nu)_\nu$ on the ray class group $\mathcal{C}l(\mathfrak{p}^\infty)$, which is tempered and whose order is directly related to the slope of the pair (π, σ) . In the slope 0 case, i.e. when the pair (π, σ) is ordinary at \mathfrak{p} , then each μ^ν is indeed a \mathfrak{p} -adic measure (cf. Theorems 4.3 and 4.4). In any case μ has the property that for certain “periods” $\Omega^{\mathrm{sign}(\chi)(-1)^\nu}(\frac{1}{2} + \nu) \in \mathbf{C}$ we have

$$\int_{\mathcal{C}l(\mathfrak{p}^\infty)} \chi d\mu^\nu = \Omega^{\mathrm{sign}(\chi)(-1)^\nu}(\frac{1}{2} + \nu) \cdot \hat{\kappa}(\mathfrak{f}) \cdot G(\chi)^{\frac{n(n-1)}{2}} \cdot L(\frac{1}{2} + \nu, (\pi \times \sigma) \otimes \chi)$$

for any Hecke character χ of finite order with non-trivial conductor $\mathfrak{f} \mid \mathfrak{p}^\infty$, and any integer ν such that $\frac{1}{2} + \nu$ is critical for $L(s, \pi \times \sigma)$ in the sense of [Del79]. Strictly speaking μ and the above interpolation formula depend on an (ordered) choice of Hecke roots for π and σ at \mathfrak{p} . See Theorem 4.5 below for the precise statement and the definition of the quantities involved.

As for today, the periods $\Omega^{\text{sign}(\chi)}(\frac{1}{2} + \nu)$ are not known to be non-zero if $n \geq 4$. However these periods have natural factorizations into local constituents, which reduces the problem of non-vanishing in the totally real case to $k = \mathbf{Q}$.

Our results overcome the restrictions on class numbers faced in [Jan11]. Furthermore we give a purely cohomological construction of the distribution and proof a functional equation for the resulting multi-valued \mathfrak{p} -adic L -function (cf. Theorem 5.4 below). The proof is however more involved than in the classically known case $n = 2$. We point out that we have no restriction on the characteristic of \mathfrak{p} . In particular our results cover characteristic 2 as well.

The different components of this \mathfrak{p} -adic L -function should be related to each other by a Manin's trick type argument. In our case however the involved finite-dimensional representations (i.e. the coefficients of cohomology) are much more involved, and up to present we were unable to generalize Manin's argument to this setting.

Although we don't use automorphic symbols here explicitly, this work was influenced by [Dim11], where the case $n = 2$ is treated with respect to additional non-abelian variables and thereby a strong connection between \mathfrak{p} -adic L -functions and the corresponding Galois deformations is established.

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Notation

If R/S is an extension of (commutative) rings, we set $M_R := R \otimes_S M$ for any S -module M . For a set M we write $\#M$ for its cardinality. If a family $(M_i)_{i \in I}$ of sets is given, let

$$\bigsqcup_{i \in I} M_i$$

denote the disjoint union of the M_i , $i \in I$. We use the notation

$$[H : H'] = \frac{[H : H \cap H']}{[H' : H \cap H']} \in \mathbf{Q}^\times$$

for the generalized index of two commensurable subgroups H, H' of a group G .

Let G be a topological group. Then G^0 denotes the connected component of the unit $1 \in G$, the same notation applies to algebraic groups with respect to the Zariski topology.

Denote by G^{der} the commutator group of a linear algebraic group G and by G^{ad} the adjoint group respectively. So G^{ad} is the image of G under the adjoint representation $\text{Ad} : G \rightarrow \text{Lie}(G)$. Here and in the sequel $\text{Lie}(G)$ is the Lie algebra of G . The differential of a morphism $f : G \rightarrow H$ of linear algebraic (or of Lie) groups is denoted by $L(f)$. $\mathcal{R}(G)$ (resp. $\mathcal{R}_{\text{u}}(G)$) is the (unipotent)

radical of G^0 . If G is defined over a number field k , we usually \mathcal{X} denotes a (connected) symmetric space for $G(k_{\mathbf{R}})$, and the superscripts ‘der’ and ‘ad’ are used to distinguish the former for $G, G^{\text{der}}, G^{\text{ad}}$.

For a global field k , we write \mathcal{O}_k for its ring of integers. We write $\mathfrak{N}(\mathfrak{a}) = [\mathcal{O}_k : \mathfrak{a}]$ for any fractional ideal $0 \neq \mathfrak{a} \subseteq k$. We denote by $k_{\mathfrak{p}}$ the completion of k at the place \mathfrak{p} and by $\mathcal{O}_{k,\mathfrak{p}}$ its valuation ring. Usually p is its residual characteristic. We write \mathbf{A}_k resp. $\mathbf{A}_k^{(\infty)}$ for the ring of (finite) adèles over k . For a place v of k we let $\mathbf{A}_k^{(\infty v)}$ denote the ring of finite adèles with the ∞ - and v -component removed.

For a quasi-character $\chi : k_{\mathfrak{p}}^{\times} \rightarrow \mathbf{C}^{\times}$ of conductor $\mathfrak{f} = \mathcal{O}_{k,\mathfrak{p}} f$ and an additive unramified character $\psi : k \rightarrow \mathbf{C}^{\times}$ we fix the ψ -Gauß sum as

$$G(\chi) := \sum_{a \pmod{\mathfrak{f}}} \chi\left(\frac{a}{f}\right) \psi\left(\frac{a}{f}\right).$$

This is independent of f and differs from the notion in [KMS00, Sch01, Jan11] by the factor $\chi(f)$. This notion of Gauß sum naturally globalizes. We define

$$t_{(f)} := \text{diag}(f^{n-1}, f^{n-2}, \dots, 1) \in \text{GL}_n(k_{\mathfrak{p}}).$$

Write w_n for the longest element of the Weyl group in GL_n (realized as permutation matrices). Define

$$h^{(1)} := \begin{pmatrix} & & & 1 \\ & & & \vdots \\ & w_{n-1} & & \vdots \\ & & & \vdots \\ 0 & \dots & 0 & 1 \end{pmatrix} \in \text{GL}_n(\mathbf{Z}),$$

and for any $f \in k_{\mathfrak{p}}^{\times}$ set

$$h^{(f)} := t_{(f)}^{-1} \cdot h^{(1)} \cdot t_{(f)} \in \text{GL}_n(k_{\mathfrak{p}}).$$

Throughout the paper we fix the diagonal embedding $j : \text{GL}_{n-1} \rightarrow \text{GL}_n$ by

$$g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}.$$

1 Hecke algebras and Hecke relations

In this section we review the interrelation of the Hecke algebra of full level and the Hecke algebra of Iwahori level. We need to extend the previous study in [Sch93, KMS00, Sch01, Jan11] slightly, as due to our possibly non-trivial central characters the action of the center matters.

1.1 Hecke algebras of finite level

For the theory of parabolic Hecke algebras we refer to [Gri92] and for an overview of what we use see [KMS00, Jan11].

For any Hecke pair (R, S) we define the free \mathbf{Z} -module $\mathcal{H}_{\mathbf{Z}}(R, S)$ over the set of all double cosets RsR , which naturally embeds into the free \mathbf{Z} -module $\mathcal{R}_{\mathbf{Z}}(R, S)$ over the set of the right cosets sR , $s \in S$, by the coset decomposition

$$RsR = \bigsqcup_i s_i R \mapsto \sum_i s_i R.$$

We identify $\mathcal{H}_{\mathbf{Z}}(R, S)$ with its image under this embedding, i.e. $\mathcal{H}_{\mathbf{Z}}(R, S)$ is the \mathbf{Z} -module of R -invariants under the action

$$R \times \mathcal{R}_{\mathbf{Z}}(R, S) \rightarrow \mathcal{R}_{\mathbf{Z}}(R, S), \quad (r, sR) \mapsto rsR.$$

Furthermore $\mathcal{H}_{\mathbf{Z}}(R, S)$ admits a structure of an associative \mathbf{Z} -algebra with the multiplication

$$\left(\sum_i s_i R \right) \cdot \left(\sum_j t_j R \right) := \sum_{i,j} s_i t_j R.$$

This algebra is unitary if and only if $R \cap S \neq \emptyset$. For any commutative ring A we set

$$\mathcal{H}_A(R, S) := \mathcal{H}_{\mathbf{Z}}(R, S) \otimes_{\mathbf{Z}} A.$$

$\mathcal{H}_A(R, S)$ is an associative algebra over A . We define the *Hecke algebra* of the pair (R, S) by $\mathcal{H}(R, S) := \mathcal{H}_{\mathbf{C}}(R, S)$.

For a locally compact topological group G and an compact open subgroup $K \leq G$ the module $\mathcal{R}_A(K, G)$ may be interpreted as the A -module of locally constant right K -invariant mappings $f : G \rightarrow A$ with compact support and $\mathcal{H}_A(K, G)$ is just the submodule of left K -invariant mappings. In this language multiplication is given by convolution

$$\alpha * \beta : x \mapsto \int_G \alpha(g) \beta(xg^{-1}) dg,$$

where dg is the right invariant Haar measure on G which assigns measure 1 to K . This integral is eventually a finite sum with integer coefficients, hence this interpretation is valid even without assuming $A \subseteq \mathbf{C}$.

All Hecke algebras we consider arise in this topological context. We have the elementary

Proposition 1.1. *Let G denote a locally compact group, $H \leq G$ a closed subgroup and let $K \leq G$ be a compact open subgroup such that $L = H \cap K$ and $HK = G$. Then the restriction*

$$\alpha \mapsto \alpha|_H$$

defines a monomorphism $\mathcal{H}_A(K, G) \rightarrow \mathcal{H}_A(L, H)$ of A -algebras.

1.2 The standard Hecke algebra

Fix a global field k and a finite place \mathfrak{p} of k . For the standard Hecke algebra $\mathcal{H}_{\mathbf{C}}(\mathrm{GL}_n(\mathcal{O}_{k,\mathfrak{p}}), \mathrm{GL}_n(k_{\mathfrak{p}}))$ at \mathfrak{p} , i.e. $K = \mathrm{GL}_n(\mathcal{O}_{k,\mathfrak{p}})$, and $G = \mathrm{GL}_n(k_{\mathfrak{p}})$ we have the Satake isomorphism

$$\mathcal{S} : \mathcal{H}_{\mathbf{C}}(K, G) \rightarrow \mathbf{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]^{S_n},$$

$$T_{\nu} \mapsto \mathfrak{N}(\mathfrak{p})^{\frac{\nu(\nu+1)}{2}} \cdot \sigma_{\nu}(X_1, \dots, X_n), \quad (0 \leq \nu \leq n)$$

where S_n is the symmetric group, permutating the X_i , and

$$T_{\nu} := K \begin{pmatrix} \mathbf{1}_{n-\nu} & 0 \\ 0 & \varpi \cdot \mathbf{1}_{\nu} \end{pmatrix} K$$

is independent of the choice of a prime ϖ . Furthermore σ_{ν} is the elementary symmetric polynomial of degree ν in X_1, \dots, X_n , cf. [Tam63, Sat63].

1.3 The Hecke algebras of Iwahori level

For an integer $r > 0$ and fix the Iwahori subgroup $K_{I^{(r)}} \subseteq K$ of level \mathfrak{p}^r as the subgroup of matrices becoming upper triangular mod \mathfrak{p}^r . As before $B := B_n(k_{\mathfrak{p}})$ denotes the standard Borel subgroup of G . It is easy to see that $I^{(r)} := BK_{I^{(r)}}$ is a closed subgroup of G , as $BI^{(r)} = I^{(r)}B$.

We have $G = I^{(r)}K$ due to the Iwasawa decomposition (see below), and furthermore $I^{(r)} \cap K = K_{I^{(r)}}$. Hence the above Proposition applies and shows that we have a canonical inclusion

$$\mathcal{H}_G := \mathcal{H}_{\mathbf{Q}}(K, G) \rightarrow \mathcal{H}_{\mathbf{Q}}(K_{I^{(r)}}, I^{(r)})$$

of the standard Hecke algebra into the Hecke algebra of Iwahori level. We will study the latter Hecke algebra by means of the parabolic Hecke algebra.

1.4 The parabolic Hecke algebra

We define $K_B := B \cap K = B_n(\mathcal{O}_{k,\mathfrak{p}})$ and the *parabolic Hecke algebra* as $\mathcal{H}_B := \mathcal{H}_{\mathbf{Q}}(K_B, B)$. Then Iwasawa decomposition [IM65], Proposition 2.33, [Sat63], section 8.2, guarantees that the hypothesis of Proposition 1.1 is fulfilled and we see that \mathcal{H}_B is a ring extension of \mathcal{H}_G , with respect to the explicit embedding $\epsilon : \mathcal{H}_G \rightarrow \mathcal{H}_B$ given by

$$\sum_i a_i \cdot g_i K \mapsto \sum_i a_i \cdot g_i K_B,$$

where $g_i \in B$. This embedding factors over the Hecke algebra of Iwahori level.

In general \mathcal{H}_B is a huge non-commutative algebra, containing a well behaved commutative subalgebra that was studied by Gritsenko. This will help us to understand (a corresponding commutative subalgebra of) the Hecke algebra of Iwahori level.

1.5 Decomposition of Hecke polynomials

We restrict our attention to the finitely generated subalgebra \mathcal{H}_I of $H_{\mathbf{Q}}(K_{I^{(r)}}, I^{(r)})$ generated by

$$U_i := K_{I^{(r)}} \begin{pmatrix} \mathbf{1}_{i-1} & 0 & 0 \\ 0 & \varpi & 0 \\ 0 & 0 & \mathbf{1}_{n-i} \end{pmatrix} K_{I^{(r)}},$$

which commute in \mathcal{H}_B by [Gri92, Lemma 2] and hence $\mathcal{H}_{I^{(r)}}$ is commutative. This algebra contains \mathcal{H}_G and following [Gri92, Theorem 2] we have over $\mathcal{H}_{I^{(r)}}$ a decomposition of the Hecke polynomial

$$H_{\mathbf{p}}(X) := \sum_{\nu=0}^n (-1)^{\nu} \mathfrak{N}(\mathbf{p})^{\frac{(\nu-1)\nu}{2}} T_{\nu} X^{n-\nu} \in \mathcal{H}_G(X)$$

into linear factors

$$H_{\mathbf{p}}(X) = \prod_{i=1}^n (X - U_i). \quad (1)$$

The unit element of $\mathcal{H}_{I^{(r)}}$ is

$$V_{\mathbf{p},0} := K_{I^{(r)}} \mathbf{1}_n K_{I^{(r)}}$$

We define for $1 \leq \nu \leq n$ the operators

$$V_{\mathbf{p},\nu} := \mathfrak{N}(\mathbf{p})^{-\frac{(\nu-1)\nu}{2}} \cdot U_1 U_2 \cdots U_{\nu} \in \mathcal{H}_{I^{(r)}}$$

and

$$V_{\mathbf{p}} := \prod_{\nu=1}^{n-1} V_{\mathbf{p},\nu} \in \mathcal{H}_{I^{(r)}}.$$

We also introduce

$$V'_{\mathbf{p}} := V_{\mathbf{p},n} \cdot V_{\mathbf{p}} = \prod_{\nu=1}^n V_{\mathbf{p},\nu} \in \mathcal{H}_{I^{(r)}}$$

Lemma 1.2. *We have for $0 \leq \nu \leq n$*

$$V_{\mathbf{p},\nu} = K_{I^{(r)}} \begin{pmatrix} \varpi \cdot \mathbf{1}_{\nu} & 0 \\ 0 & \mathbf{1}_{n-\nu} \end{pmatrix} K_{I^{(r)}} = \bigsqcup_A \begin{pmatrix} \varpi \cdot \mathbf{1}_{\nu} & A \\ 0 & \mathbf{1}_{n-\nu} \end{pmatrix} K_{I^{(r)}},$$

where $A \in \mathcal{O}_{k,\mathbf{p}}^{\nu \times n-\nu}$ runs through a system of representatives modulo \mathbf{p} . Furthermore the Hecke operators $V_{\mathbf{p},\nu}$ commute for $0 \leq \nu \leq n$ and

$$V_{\mathbf{p}} = K_{I^{(r)}} t_{(\varpi)} K_{I^{(r)}} = \bigsqcup_u u t_{(\varpi)} K_{I^{(r)}},$$

where u runs through a system of representatives of $U_n(\mathcal{O}_{k,\mathbf{p}})/t_{(\varpi)} U_n(\mathcal{O}_{k,\mathbf{p}}) t_{(\varpi)}^{-1}$. Furthermore

$$V'_{\mathbf{p}} = K_{I^{(r)}} \varpi t_{(\varpi)} K_{I^{(r)}} = \bigsqcup_u u \varpi t_{(\varpi)} K_{I^{(r)}},$$

Proof. The first part of the lemma is the same as [KMS00, Lemma 4.1], at least when $0 \leq \nu < n$. The case $\nu = n$ as well as the last part follow from the identity

$$T_n = V_{\mathfrak{p},n},$$

which is an immediate consequence of Gritsenko's factorization (1). \square

1.6 The projection formula

Let $\underline{\lambda} = (\lambda_1, \dots, \lambda_m) \in E^m$ for $0 \leq m \leq n$. We define the $\mathcal{H}_{I(r)}$ submodule $\mathcal{M}^{\underline{\lambda}}$ of \mathcal{M} consisting of all $\psi \in \mathcal{M}$ such that

$$\forall \nu = 1, 2, \dots, m : \quad H_{\mathfrak{p}}(\lambda_{\nu}) \cdot \psi = 0. \quad (2)$$

Furthermore we set

$$\eta_{\nu} := \mathfrak{N}(\mathfrak{p})^{-\frac{\nu(\nu-1)}{2}} \prod_{i=1}^{\nu} \lambda_i$$

for $1 \leq \nu \leq m$. We denote by $\mathcal{M}_{\underline{\lambda}}$ the $\mathcal{H}_{I(r)}$ submodule of $\mathcal{M}^{\underline{\lambda}}$ consisting of vectors $\psi \in \mathcal{M}$ that are simultaneous eigen functions for $V_{\mathfrak{p},1}, \dots, V_{\mathfrak{p},m}$ with eigen value η_{ν} , i.e. the subspace of ψ satisfying

$$V_{\mathfrak{p},\nu} \cdot \psi = \eta_{\nu} \cdot \psi$$

for $1 \leq \nu \leq m$.

Proposition 1.3. *Let \mathcal{M} be a $\mathcal{H}_{I(r)}$ -module over a field E . Let $\underline{\lambda} = (\lambda_1, \dots, \lambda_m) \in E^m$ for $0 \leq m \leq n$. Then the map*

$$\Pi_{\underline{\lambda}}^0 : \psi \mapsto \prod_{i=1}^m \prod_{\substack{j=1 \\ j \neq i}}^n (\lambda_i \mathfrak{N}(\mathfrak{p})^{1-j} V_{\mathfrak{p},j-1} - V_{\mathfrak{p},j}) \cdot \psi$$

is a well defined $\mathcal{H}_{I(r)}$ -module map

$$\Pi_{\underline{\lambda}}^0 : \mathcal{M}^{\underline{\lambda}} \rightarrow \mathcal{M}_{\underline{\lambda}}.$$

Proof. As the Hecke operators U_1, \dots, U_n commute with $V_{\mathfrak{p},0}, \dots, V_{\mathfrak{p},m}$, we see that $\Pi_{\underline{\lambda}}^0$ is indeed an endomorphism of $\mathcal{M}^{\underline{\lambda}}$.

That $\Pi_{\underline{\lambda}}^0$ is a well defined $\mathcal{H}_{I(r)}$ -module homomorphism $\mathcal{M}^{\underline{\lambda}} \rightarrow \mathcal{M}_{\underline{\lambda}}$ was proven for $k = \mathbf{Q}$, $m = n - 1$ and $r = 1$ in [KMS00, Proposition 4.2]. The proof given there eventually shows the slightly more general statement for any k , m and r . \square

Proposition 1.4. *Let \mathcal{M} be a $\mathcal{H}_{I(r)}$ -module over a field E . Let $\underline{\lambda} = (\lambda_1, \dots, \lambda_m) \in E^m$ with pairwise distinct non-zero $\lambda_1, \dots, \lambda_m$ for $0 \leq m \leq n$. Then the map*

$$\Pi_{\underline{\lambda}} : \psi \mapsto \prod_{i=1}^m \prod_{\substack{j=1 \\ j \neq i}}^n \frac{\lambda_i \mathfrak{N}(\mathfrak{p})^{1-j} V_{\mathfrak{p},j-1} - V_{\mathfrak{p},j}}{\lambda_i \cdot \mathfrak{N}(\mathfrak{p})^{1-j} \cdot \eta_{j-1} - \eta_j} \cdot \psi$$

is a well defined projection

$$\Pi_{\underline{\lambda}} : \mathcal{M}^{\underline{\lambda}} \rightarrow \mathcal{M}_{\underline{\lambda}}.$$

Proof. Due to Proposition (1.3) it remains only to show that $\Pi_{\underline{\lambda}}$ is indeed a projection, i.e. induces the identity on $\mathcal{M}_{\underline{\lambda}}$. We proof this by induction on m , the case $m = 0$ being clear. Assume that $m > 0$. Set $\underline{\lambda}' := (\lambda_1, \dots, \lambda_{m-1})$ and by our induction hypothesis $\Pi_{\underline{\lambda}'}$ induces the identity on $\mathcal{M}^{\underline{\lambda}'}$. Pick any $\psi \in \mathcal{M}^{\underline{\lambda}}$. Then ψ lies in $\mathcal{M}^{\underline{\lambda}'}$ and therefore

$$\begin{aligned} \Pi_{\underline{\lambda}}(\psi) &= \prod_{\substack{j=1 \\ j \neq m}}^n \frac{\lambda_m \mathfrak{N}(\mathfrak{p})^{1-j} V_{\mathfrak{p},j-1} - V_{\mathfrak{p},j}}{\lambda_m \cdot \mathfrak{N}(\mathfrak{p})^{1-j} \cdot \eta_{j-1} - \eta_j} \cdot \Pi_{\underline{\lambda}'}(\psi) = \prod_{\substack{j=1 \\ j \neq m}}^n \frac{\lambda_m \mathfrak{N}(\mathfrak{p})^{1-j} V_{\mathfrak{p},j-1} - V_{\mathfrak{p},j}}{\lambda_m \cdot \mathfrak{N}(\mathfrak{p})^{1-j} \cdot \eta_{j-1} - \eta_j} \cdot \psi = \\ &= \prod_{\substack{j=1 \\ j \neq n}}^n \frac{\lambda_m \cdot \mathfrak{N}(\mathfrak{p})^{1-j} \cdot \eta_{j-1} - \eta_j}{\lambda_m \cdot \mathfrak{N}(\mathfrak{p})^{1-j} \cdot \eta_{j-1} - \eta_j} \cdot \psi = \psi, \end{aligned}$$

because ψ is an eigen vector for $V_{\mathfrak{p}}$, with eigen value η_m . \square

2 The Birch Lemma

In this section we generalize the Birch Lemma of [Jan11] to pairs (π, σ) which are allowed to be of level $K_{I(r)}$ at \mathfrak{p} and are minimal among \mathfrak{p} -power twists. We also renormalize the Birch Lemma, which enables us to overcome the class number restriction encountered in [Jan11].

2.1 The local Zeta integral

We use the notation of [Jan11, Section 2] in the following modified setting. Let $\chi : F^\times \rightarrow \mathbf{C}^\times$ be a character of a local field F of non-trivial conductor \mathfrak{f}_χ generated by $f_\chi = \varpi^s$. We fix another element $f = \varpi^r \in \mathcal{O}_F$ with $r \geq s$ and write $I_n^{(r)}$ for the Iwahori subgroup of $\mathrm{GL}_n(\mathcal{O}_F)$ of level f .

All quantities that are defined relative to f retain their meaning, i.e. the matrices $A_n, \tilde{A}_n, B_n, C_n, D_n, E_n, \phi_n$ are all defined with respect to $f = \varpi^r$, as are the groups $J_{l,n}$ and $\overline{T}_{l,n}$.

We define $\mathfrak{R}_{l,n}$ and its variants as before, i.e. via $I_n = I_n^{(1)}$. Assume as before that $l \geq 2n$. We have

$$J_{l,n} \subseteq I_n^{(r)} \cap w_n D_n^{-1} I_n^{(r)} D_n w_n,$$

generalizing equation (6) of loc. cit..

For any $\delta \in \mathbf{Z}$ we define

$$\begin{aligned} j_\delta : \mathrm{GL}_n(F) &\rightarrow \mathrm{GL}_{n+1}(F), \\ g &\mapsto \begin{pmatrix} g & 0 \\ 0 & \varpi^\delta \end{pmatrix}, \end{aligned}$$

and

$$\lambda_n^\delta(g) := \lambda_n(\varpi^{-\delta} \cdot g).$$

We need the following generalized statement of Lemma 2.6 of loc. cit.

Lemma 2.1. *For any $I_{n+1}^{(r)}$ - resp. $I_n^{(r)}$ -invariant ψ - resp. ψ^{-1} -Whittaker functions w and v on $\mathrm{GL}_{n+1}(F)$ resp. $\mathrm{GL}_n(F)$ and any $\delta \in \mathbf{Z}$ we have*

$$\begin{aligned} w(j_\delta(g)C_{n+1} \cdot D_{n+1}w_{n+1})v(g) = \\ \psi(\lambda_n^\delta(gB_n))w(j_\delta(gB_n \cdot D_nw_n))v(gB_n). \end{aligned} \quad (3)$$

Proof. First observe that all relations in the proof of Lemma 2.6 of loc. cit. eventually are valid modulo $I_{n+1}^{(r)}$ as well. This shows in particular the case $\delta = 0$. The general case may be reduced to this case as follows. We have

$$j_\delta(g) = \Delta_\delta \cdot j_0(g),$$

where

$$\Delta_\delta := \mathrm{diag}(1, \dots, 1, \varpi^\delta).$$

From the aforementioned proof of Lemma 2.6 in loc. cit. we know that

$$w(\Delta_\delta u \Delta_\delta^{-1} j_\delta(g) C_{n+1} D_{n+1} w_{n+1}) = w(j_\delta(g B_n D_n w_n),$$

which together with

$$\Psi(\Delta_\delta u \Delta_\delta^{-1})^{-1} = \Psi(\lambda_n(\varpi^{-\delta} \cdot g B_n))$$

concludes the proof. \square

Set $\delta := r - s$ and $d_\delta := (\delta \cdot (n+1-i))_{1 \leq i \leq n} \in \mathbf{Z}^n$. For any integer $\delta' \in \mathbf{Z}$ we write $(\delta') \in \mathbf{Z}^n$ for the vector which has δ' in each components.

Lemma 2.2. *Let w and v be Iwahori invariant ψ - (resp. ψ^{-1} -) Whittaker functions on $\mathrm{GL}_n(F)$. For any $n \geq 0$, $e \in \mathbf{Z}^n$, $\omega \in W_n$, $l \geq \max\{2n, n - e_1/\nu_{\mathfrak{p}}(f), \dots, n - e_n/\nu_{\mathfrak{p}}(f)\}$ and $\delta' \in \mathbf{Z}$ we have*

$$\begin{aligned} \sum_{g \in \varpi^e \omega \mathfrak{H}_{l,n}^\omega} \psi(\lambda_n^{\delta'}(g)) \cdot w(g \cdot D_n w_n) \cdot v(g) \cdot \chi(\det(g)) \cdot \|\det(g)\|^s = \\ \begin{cases} \mathfrak{N}(\mathfrak{f})^{\frac{(l-2n+1)n(n+1)}{2} + \frac{1}{2} \sum_{\nu=1}^n 5\nu^2 - 3\nu} \mathfrak{N}(\mathfrak{f}_\chi)^{-\frac{n(n+1)}{2}} (\chi(f_\chi) G(\chi))^{\frac{n(n+1)}{2}} \\ \quad \cdot w(\varpi^{\delta+\delta'} t_{(\varpi)}^\delta) v(\varpi^{\delta+\delta'} t_{(\varpi)}^\delta) \chi(\det \varpi^{\delta+\delta'} t_{(\varpi)}^\delta) \left\| \det \varpi^{\delta+\delta'} t_{(\varpi)}^\delta \right\|^s, \\ \quad \text{for } \omega = \mathbf{1}_n \text{ and } e = d_\delta + (\delta'), \\ 0, \quad \text{otherwise.} \end{cases} \end{aligned}$$

Proof. We closely follow the proof of Lemma 2.7 in loc. cit., and only briefly indicate the necessary modifications here. We proceed again by induction, the the partial sums $Z(r)$ being defined as before, using $\lambda_n^{\delta'}$ instead of λ_n .

In the argument of v the parameter r might not be dropped in our setting in the course of the proof. However, this modification is straightforward and we cease to indicate it again.

We have the generalized relation

$$\varpi^e \omega^\gamma r \cdot D_n w_n \in \varpi^e \omega r \cdot D_n w_n \cdot I_n^{(r)},$$

which implies that, up to the abovementioned missing r in the argument of the Whittaker function v and the replacement of λ_n by $\lambda_n^{\delta'}$, the formula for $Z(r)$ in the bottom of page 20 of loc. cit. remains valid, and the second formula on page 21 now reads

$$\sum_{\gamma \in S} \chi(\gamma \mathbf{1}_n) \cdot \psi(\lambda_n^{\delta'}(\varpi^e \omega^\gamma r)) = \prod_{\nu=1}^n \sum_{\gamma_\nu \in (\mathcal{O}_F/\mathfrak{f}^l)^\times} \chi(\gamma_\nu) \cdot \psi\left(\varpi^{e_n - \delta'} f^{\nu-n-1} r_{\sigma(n)\nu} \cdot \gamma_\nu\right).$$

Therefore the analogue of conclusion (10) of loc. cit. here is

$$e_n \neq (n - \sigma(n)) \cdot r + \delta + \delta' \Rightarrow Z(r) = 0.$$

Hence we may assume that

$$e_n = (n - \sigma(n)) \cdot r + \delta + \delta',$$

which means that if $\sigma(n) \neq n$, then $e_n > \delta + \delta'$. This then implies

$$\left\| \varpi^{e_n - \delta'} f^{n-n-1} r_{\sigma(n)n} \cdot \gamma_n \right\| < \|f_\chi^{-1}\|,$$

yielding again

$$Z(r) = 0.$$

Therefore we can assume $\sigma(n) = n$ and $e_n = \delta + \delta'$. The implication (11) of loc. cit. is valid without change, and so we may restrict to the case $r_{1n} = f^{n-1}$ and

$$r_{n\nu} = -f^{n-\nu}, \quad 2 \leq \nu \leq n$$

as before. Due to our modified Gauß sum equation (12) of loc. cit. now reads

$$\sum_{\gamma \in S} \chi(\gamma \mathbf{1}_n) \cdot \psi(\lambda_n^{\delta'}(\varpi^e \omega^\gamma r)) = \chi(B_n) \cdot (\chi(f_\chi) G(\chi))^n \cdot \mathfrak{N}(\mathfrak{f})^{l \cdot n} \mathfrak{N}(\mathfrak{f}_\chi)^{-n}. \quad (4)$$

The rest of the proof remains valid with the obvious changes that are implied by $e_n = \delta + \delta'$ and the distinction between f_χ and f , thanks to the validity of

(3), which allows for the same inductive argument to remain intact. To make this precise, the relation (3) and the identity (4) together imply that

$$\sum_{r \in \mathfrak{N}_{l,n}^\omega} Z(r) = (\chi(f_\chi)G(\chi))^n \cdot \mathfrak{N}(\mathfrak{f})^{l \cdot n} \mathfrak{N}(\mathfrak{f}_\chi)^{-n} \left\| \varpi^{\delta+\delta'} \right\|^s \chi(\varpi^{\delta+\delta'}) \cdot \sum_{\tilde{r} \in \mathfrak{N}_{l,n-1}^\omega} \tilde{Z}(\tilde{r}),$$

where the partial sum $\tilde{Z}(\tilde{r})$ is defined mutatis mutandis as $Z(r)$ for the truncated parameters \tilde{e} , $\tilde{\omega}$ and \tilde{r} for $\mathrm{GL}_{n-1}(F)$ and the map $\lambda_{n-1}^{\delta+\delta'}$.

The induction hypothesis shows that those partial sums vanish whenever $\omega \neq \mathbf{1}_n$ or

$$e \neq d_\delta + (\delta + \delta').$$

Introduce the map

$$\begin{aligned} \tilde{j}_{\delta+\delta'} : \mathrm{GL}_{n-1}(F) &\rightarrow \mathrm{GL}_n(F), \\ \tilde{g} &\mapsto \begin{pmatrix} \tilde{g} & 0 \\ 0 & \varpi^{\delta+\delta'} \end{pmatrix}, \end{aligned}$$

We get for $e = d_\delta + (\delta')$ and $\omega = \mathbf{1}_n$, using the notation \tilde{d}_δ for the obvious truncation,

$$\begin{aligned} \sum_{g \in \varpi^{d_\delta+(\delta')} \mathfrak{N}_{l,n}^{\mathbf{1}_n}} \psi(\lambda_n^{\delta'}(g)) w(g \cdot D_n w_n) v(g) \chi(g) \|\det(g)\|^s &= \mathfrak{N}(\mathfrak{f})^{-\frac{n(n-1)}{2}} \cdot \sum_{r \in \mathfrak{N}_{l,n}^\omega} Z(r) = \\ &= (\chi(f_\chi)G(\chi))^n \cdot \mathfrak{N}(\mathfrak{f})^{l \cdot n} \cdot \mathfrak{N}(\mathfrak{f}_\chi)^{-n} \cdot \left\| \varpi^{\delta+\delta'} \right\|^s \cdot \chi(\varpi^{\delta+\delta'}). \\ \mathfrak{N}(\mathfrak{f})^{\frac{(n-1)(n-2)}{2}} \cdot \sum_{\tilde{g} \in \varpi^{\tilde{d}_\delta+(\delta+\delta')} \mathfrak{N}_{l,n-1}^{\mathbf{1}_{n-1}}} \psi(\lambda_{n-1}^{\delta+\delta'}(\tilde{g})) w(\tilde{j}(\tilde{g} \cdot D_{n-1} w_{n-1})) v(\tilde{j}(\tilde{g})) \chi(\tilde{g}) \|\det(\tilde{g})\|^s &= \\ &= (\chi(f_\chi)G(\chi))^n \cdot \mathfrak{N}(\mathfrak{f})^{l \cdot n} \cdot \mathfrak{N}(\mathfrak{f}_\chi)^{-n} \cdot \left\| \varpi^{\delta+\delta'} \right\|^s \cdot \chi(\varpi^{\delta+\delta'}) \cdot \mathfrak{N}(\mathfrak{f}_\chi)^{-\frac{n(n-1)}{2}} \cdot \\ &\quad \mathfrak{N}(\mathfrak{f})^{\frac{(n-1)(n-2)}{2} + \frac{(l-2)(n-1)+1)n(n-1)}{2} + \frac{1}{2} \sum_{\nu=1}^{n-1} 5\nu^2 - 3\nu}. \\ (\chi(f_\chi)G(\chi))^{\frac{n(n-1)}{2}} \cdot w(\varpi^{d_\delta+(\delta')}) v(\varpi^{d_\delta+(\delta')}) \chi(\varpi^{\tilde{d}_\delta+(\delta+\delta')}) \left\| \det \varpi^{\tilde{d}_\delta+(\delta+\delta')} \right\|^s &, \end{aligned}$$

by our induction hypothesis, and the claim follows. \square

Theorem 2.3. *Let w and v be ψ - (resp. ψ^{-1} -) Whittaker functions on $\mathrm{GL}_{n+1}(F)$ resp. $\mathrm{GL}_n(F)$, Iwahori invariant of level \mathfrak{f} , and $\chi : F^\times \rightarrow \mathbf{C}^\times$ a character with conductor $1 \neq \mathfrak{f}_\chi \mid \mathfrak{f}$. Then*

$$\begin{aligned} \int_{U_n(F) \backslash \mathrm{GL}_n(F)} w \left(j(g) \cdot t_{(ff_\chi^{-1})} \cdot h^{(f)} \right) v(g \cdot f f_\chi^{-1} t_{(ff_\chi^{-1})}) \chi(\det(g)) \|\det(g)\|^{s-\frac{1}{2}} dg &= \\ \prod_{\nu=1}^n (1 - \mathfrak{N}(\mathfrak{p})^{-\nu})^{-1} \cdot \mathfrak{N}(\mathfrak{f})^{-\frac{(n+1)n(n-1)}{6}} \cdot \mathfrak{N}(\mathfrak{f}_\chi)^{-\frac{n(n+1)}{2}} \cdot (\chi(f_\chi)G(\chi))^{\frac{n(n+1)}{2}} &\cdot \\ w(t_{(ff_\chi^{-1})}) \cdot v(f f_\chi^{-1} \cdot t_{(ff_\chi^{-1})}) &. \end{aligned}$$

Proof. The proof proceeds as the proofs of Theorem 2.1 and Corollary 2.8 of [Jan11], using Lemma 2.2, via the substitution $g \mapsto g \cdot \varpi^d$. \square

2.2 The global Zeta integral

Choose a global field k , i.e. a finite extension of \mathbf{Q} or $\mathbf{F}_p(T)$ and fix an additive character $\psi : k \backslash \mathbf{A}_k \rightarrow \mathbf{C}$ with a local factorization as in [Jan11, section 3]. Let π and σ be irreducible cuspidal automorphic representations of $\mathrm{GL}_n(\mathbf{A}_k)$ and $\mathrm{GL}_{n-1}(\mathbf{A}_k)$ respectively. Note that π and σ are always generic [Sha74]. By S_∞ we denote the set of infinite places of k . Let S denote the set of finite places where π or σ ramifies. Furthermore fix a finite place \mathfrak{p} such that $\pi_{\mathfrak{p}}$ and $\sigma_{\mathfrak{p}}$ possess non-zero $I_n^{(r)}$ resp. $I_{n-1}^{(r)}$ -invariant vectors for some fixed $r \geq 0$.

For an overview of the theory of Rankin-Selberg L -function $L(s, \pi \times \sigma)$ as developed in [JPSS79a, JPSS79b, JPSS83, JS90, CPS94, CPS04] the reader might consult [KMS00, Jan11] for all facts we use.

At any finite place \mathfrak{q} of k we pick a *good tensor* $t_{\mathfrak{q}}^0 \in \mathcal{W}(\pi_{\mathfrak{q}}, \psi_{\mathfrak{q}}) \otimes \mathcal{W}(\sigma_{\mathfrak{q}}, \psi_{\mathfrak{q}}^{-1})$ in the local Whittaker spaces such that the local Euler factor at \mathfrak{q} is given by the corresponding local zeta integral for $t_{\mathfrak{q}}^0$, ie.

$$L(s, \pi_{\mathfrak{q}} \times \sigma_{\mathfrak{q}}) = \Psi(t_{\mathfrak{q}}^0, s),$$

where the right hand side denotes the local Rankin-Selberg zeta integral (or a finite linear combination of those) as in [JPSS83]. We suppose that $t_{\mathfrak{q}}^0 = w^0 \otimes v^0$ for class-1 w^0 and v^0 whenever possible (ie. when π and σ are spherical at \mathfrak{q}). By Shintani's explicit formula [Shi76] this Euler factor is given explicitly by

$$L(s, \pi_{\mathfrak{q}} \times \sigma_{\mathfrak{q}}) = \det(\mathbf{1}_{n(n-1)} - \mathfrak{N}(\mathfrak{q})^{-s} A_{\pi_{\mathfrak{q}}} \otimes A_{\sigma_{\mathfrak{q}}})^{-1},$$

for any place $\mathfrak{q} \notin S \cup S_\infty$, where $A_{\pi_{\mathfrak{q}}}$ and $A_{\sigma_{\mathfrak{q}}}$ denote the corresponding Satake parameters.

Now pick any archimedean Whittaker functions (corresponding to K -finite vectors) $(w_{\mathfrak{q}}, v_{\mathfrak{q}})$ for $\mathfrak{q} \in S_\infty$ and form a pair $(w, v) \in \mathcal{W}_0(\pi, \psi) \times \mathcal{W}_0(\sigma, \psi^{-1})$ of global Whittaker functions with factorizations $w = \bigotimes_{\mathfrak{q}} w_{\mathfrak{q}}$, $v = \bigotimes_{\mathfrak{q}} v_{\mathfrak{q}}$. By Fourier transform we have associated automorphic forms ϕ on $\mathrm{GL}_n(\mathbf{A}_k)$ and φ on $\mathrm{GL}_{n-1}(\mathbf{A}_k)$ respectively. For $\mathrm{Re}(s) \gg 0$ the Euler product

$$\prod_{\mathfrak{q}} \Psi(w_{\mathfrak{q}}, v_{\mathfrak{q}}, s) = \int_{\mathrm{GL}_{n-1}(k) \backslash \mathrm{GL}_{n-1}(\mathbf{A}_k)} \phi(j(g)) \varphi(g) \|\det(g)\|^{s-\frac{1}{2}} dg$$

converges absolutely and has an analytic continuation to \mathbf{C} , as the right hand side is entire. Furthermore we find an entire function Ω , depending only on the Whittaker functions at infinity, such that for the global L -function

$$\Omega(s) \cdot L(s, \pi \times \sigma) = \prod_{\mathfrak{q} \in S_\infty} \Psi(w_{\mathfrak{q}}, v_{\mathfrak{q}}, s) \cdot \prod_{\mathfrak{q} \notin S_\infty} \Psi(t_{\mathfrak{q}}^0, s),$$

for $\mathrm{Re}(s) \gg 0$. Writing

$$w_\infty := \bigotimes_{\mathfrak{q} \in S_\infty} w_{\mathfrak{q}}$$

and

$$v_\infty := \bigotimes_{\mathfrak{q} \in S_\infty} v_{\mathfrak{q}}$$

we set

$$(w_\infty \otimes v_\infty) \otimes_{\mathfrak{q}} t_{\mathfrak{q}}^0 = \sum_{\iota} w_{\iota} \otimes v_{\iota},$$

where any $w_{\iota} \otimes v_{\iota}$ is a product of pure tensors, we deduce that with the corresponding associated automorphic forms $(\phi_{\iota}, \varphi_{\iota})$ we get

$$\begin{aligned} & \Omega(w_\infty \otimes v_\infty, 1)(s) \cdot L(s, \pi \times \sigma) = \\ & \sum_{\iota} \int_{\mathrm{GL}_{n-1}(k) \backslash \mathrm{GL}_{n-1}(\mathbf{A}_k)} \phi_{\iota}(j(g)) \varphi_{\iota}(g) \|\det(g)\|^{s-\frac{1}{2}} dg, \end{aligned}$$

for the entire function $\Omega(w_\infty \otimes v_\infty, 1)(s) =: \Omega(s)$, depending on our choices, which is \mathbf{C} -linear in the first argument, the second argument being reserved for a character of $\mathrm{GL}_1(k \otimes_{\mathbf{Q}} \mathbf{R})$.

In order to study the twisted L -function

$$L(s, (\pi \times \sigma) \otimes \chi) := L(s, (\pi \otimes \chi) \times \sigma)$$

for a quasi-character χ with \mathfrak{p} -power conductor \mathfrak{f}_χ we modify the local Whittaker functions at \mathfrak{p} and allow Iwahori invariant pairs only.

Theorem 2.4. *For any choice of pair of Whittaker functions (w_∞, v_∞) at infinity, and any pair $(w_{\mathfrak{p}}, v_{\mathfrak{p}})$ of $I_n^{(r)}$ - resp. $I_{n-1}^{(r)}$ -invariant Whittaker functions on $\mathrm{GL}_n(k_{\mathfrak{p}})$ and $\mathrm{GL}_{n-1}(k_{\mathfrak{p}})$ respectively, there exists an entire function $\Omega(w_\infty \otimes v_\infty, \chi_\infty)$, only depending on $(w_\infty \otimes v_\infty)$ and χ_∞ , such that for any quasi-character $\chi : k^\times \backslash \mathbf{A}_k^\times \rightarrow \mathbf{C}^\times$ with non-trivial \mathfrak{p} -power conductor $\mathfrak{f}_\chi \mid \mathfrak{f} = \mathfrak{p}^r$ we have*

$$\begin{aligned} & \Omega(w_\infty \otimes v_\infty, \chi_\infty)(s) \delta^{(r)}(w_{\mathfrak{p}} \otimes v_{\mathfrak{p}}, \chi_{\mathfrak{p}}) (\chi(f_\chi) G(\chi))^{\frac{n(n-1)}{2}} \cdot \\ & \mathfrak{N}(\mathfrak{f}_\chi)^{-\frac{n(n-1)}{2}} \cdot \mathfrak{N}(\mathfrak{f})^{-\frac{n(n-1)(n-2)}{6}} \cdot L(s, (\pi \otimes \chi) \times \sigma) = \\ & \sum_{\iota} \int_{\mathrm{GL}_{n-1}(k) \backslash \mathrm{GL}_{n-1}(\mathbf{A}_k)} \phi_{\iota} \left(j(g) t_{(ff_\chi^{-1})} h^{(f)} \right) \varphi_{\iota}(g f f_\chi^{-1} t_{(ff_\chi^{-1})}) \chi(\det(g)) \|\det(g)\|^{s-\frac{1}{2}} dg, \end{aligned}$$

where

$$\delta^{(r)}(w_{\mathfrak{p}} \otimes v_{\mathfrak{p}}, \chi_{\mathfrak{p}}) := w_{\mathfrak{p}}(t_{(ff_\chi^{-1})}) \cdot v_{\mathfrak{p}}(f f_\chi^{-1} \cdot t_{(ff_\chi^{-1})}) \cdot \prod_{\nu=1}^n (1 - \mathfrak{N}(\mathfrak{p})^{-\nu})^{-1}.$$

Proof. It is clear that to compute the twisted L -function we might choose at any infinite place \mathfrak{q} the pair of local Whittaker functions $(\chi_{\mathfrak{q}}(\det) \cdot w_{\mathfrak{q}}, v_{\mathfrak{q}})$ for even n or $(w_{\mathfrak{q}}, \chi_{\mathfrak{q}}(\det) \cdot v_{\mathfrak{q}})$ for odd n . This data will account for $\Omega(w_\infty \otimes v_\infty, \chi_\infty)$. The rest of the argument is reduced to Theorem 2.3 as in [Jan11, Proof of Theorem 3.1]. \square

Let $U_{\mathfrak{q}} := \mathbf{G}_m(\mathcal{O}_{k,\mathfrak{q}})$ for nonarchimedean \mathfrak{q} and define $U_{\mathfrak{q}} := \mathbf{G}_m(k_{\mathfrak{q}})^0$ for $\mathfrak{q} \in S_{\infty}$. For an idèle $\alpha \in \mathbf{A}_k^{\times}$ we let $C_{\mathfrak{f}}$ denote the preimage of

$$k^{\times} \backslash k^{\times} \cdot (1 + \mathfrak{f}) \cdot \prod_{\mathfrak{q} \nmid \mathfrak{f}} U_{\mathfrak{q}}$$

under the determinant map

$$\det : \mathrm{GL}_n(k) \backslash \mathrm{GL}_n(\mathbf{A}_k) \rightarrow k^{\times} \backslash \mathbf{A}_k^{\times}.$$

For any idèle $x \in \mathbf{A}_k^{\times}$ we set

$$d_{(x)} := \mathrm{diag}(x, 1, \dots, 1).$$

As a consequence of Theorem 2.4 we have

Corollary 2.5. *For any χ of finite order and conductor $\mathfrak{f}_{\chi} \mid \mathfrak{f}$ and any $\nu \in \mathbf{Z}$ we have*

$$\Omega(w_{\infty} \otimes v_{\infty}, \chi_{\infty}) \left(\frac{1}{2} + \nu \right) \delta^{(r)}(w_{\mathfrak{p}} \otimes v_{\mathfrak{p}}, \chi_{\mathfrak{p}}) (\chi(f_{\chi}) G(\chi))^{\frac{n(n-1)}{2}}.$$

$$\mathfrak{N}(\mathfrak{f}_{\chi})^{-\frac{n(n-1)}{2}} \mathfrak{N}(\mathfrak{f})^{-\frac{n(n-1)(n-2)}{6}} \cdot L\left(\frac{1}{2} + \nu, (\pi \otimes \chi) \times \sigma\right) =$$

$$\sum_{\iota, x} \chi(x) \cdot \int_{C_{\mathfrak{f}}} \phi_{\iota} \left(j(gd_{(x)}) \cdot t_{(ff_{\chi}^{-1})} \cdot h^{(f)} \right) \cdot \varphi_{\iota}(gd_{(x)} \cdot ff_{\chi}^{-1} t_{(ff_{\chi}^{-1})}) \left\| \det(gd_{(x)}) \right\|^{\nu} dg.$$

Here x runs through a system of representatives of the ray class group $k^{\times} \backslash \mathbf{A}_k^{\times} / (1 + \mathfrak{f}) \prod_{\mathfrak{q} \neq \mathfrak{p}} U_{\mathfrak{q}}$.

3 Arithmetic groups and relative Lie algebra cohomology

The historically inclined reader might consult the fundamental articles of Matsushima and Murakami [MM63, MM65]. The modern main reference is of course [BW80]. We assume here k to denote a number field. We write G_n for the restriction of scalars of GL_n in the extension k/\mathbf{Q} . Let K denote a maximal compact subgroup of $G := G_n(\mathbf{R})$ and θ the corresponding (algebraic) Cartan involution. We write $\mathfrak{g} = \mathfrak{gl}_n \otimes \mathbf{C}$ for the complexified Lie algebra of G and \mathfrak{k} for the complexified Lie algebra of K . We can identify \mathfrak{k} with the $(+1)$ -eigen space of θ acting on \mathfrak{g} , and likewise we have a (-1) -eigen space that we denote \mathfrak{p} . Then

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k},$$

i.e. \mathfrak{p} is canonically identified with $\mathfrak{g}/\mathfrak{k}$.

3.1 Relative Lie algebra cohomology

Pick a (\mathfrak{g}, K) -module (π, V) , for example the space of K -finite vectors in an automorphic representation of G_n . By the very definition of (\mathfrak{g}, K) -modules, the compatibility of the actions of \mathfrak{g} and K on V reads

$$\pi(z) \cdot \pi(g) \cdot v = \pi(\text{Ad}(z)(g)) \cdot \pi(z) \cdot v$$

for all $g \in \mathfrak{g}$, $z \in K$. Furthermore, as v is contained in a finite dimensional K -stable subspace $W \subseteq V$, we have

$$\pi(L(z)) \cdot w = L(\pi_W(z)) \cdot w$$

for all $w \in W$, where π_W denotes the representation of K on W induced by π . In other words the differential of π_W is given by $\pi_{\mathfrak{k}}$.

Consider the complex

$$C^q(\mathfrak{g}, \mathfrak{k}; V) := \text{Hom}_{\mathfrak{k}}\left(\bigwedge^q \mathfrak{p}, V\right),$$

with the differential $d : C^q \rightarrow C^{q+1}$ given by

$$\begin{aligned} df(x_0 \wedge \cdots \wedge x_q) &= \sum_i (-1)^i \cdot x_i \cdot f(x_0 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_q) + \\ &\sum_{i < j} (-1)^{i+j} f([x_i, x_j] \wedge x_0 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_q). \end{aligned}$$

This gives rise to classical relative Lie algebra cohomology denoted $H^q(\mathfrak{g}, \mathfrak{k}; V)$. If V is admissible, then this cohomology is finite dimensional, as the complex itself is finite dimensional.

3.2 (\mathfrak{g}, K) -cohomology

Now K naturally acts on \mathfrak{p} by the adjoint action and it acts naturally on V as well. So we have another complex

$$C^q(\mathfrak{g}, K; V) := \text{Hom}_K\left(\bigwedge^q \mathfrak{p}, V\right),$$

which eventually turns out to be a subcomplex of the former. To identify this subcomplex, note that $\pi_0(G)$ can be canonically identified with K/K^0 and the latter group acts naturally on the first complex (again via the adjoint representation on \mathfrak{p}). Denote by

$$i : C^q(\mathfrak{g}, K; V) \rightarrow C^q(\mathfrak{g}, \mathfrak{k}; V)$$

the canonical inclusion.

Then for any f in $C^q(\mathfrak{g}, \mathfrak{k}; V)$ and any $x \in \bigwedge^q \mathfrak{p}$ we know that $f(x)$ is contained in a finite dimensional K -stable subspace of V . Furthermore K^0 acts trivially on f and moreover

$$C^q(\mathfrak{g}, \mathfrak{k}; V) = C^q(\mathfrak{g}, K^0; V) := \text{Hom}_{K^0}(\bigwedge^q \mathfrak{p}, V).$$

So we eventually get a well defined action of $\pi_0(G)$ on this space. We conclude that for this action

$$C^q(\mathfrak{g}, K; V) = \text{Hom}_K(\bigwedge^q \mathfrak{p}, V) = C^q(\mathfrak{g}, \mathfrak{k}; V)^{\pi_0(G)}.$$

Finally for the respective cohomologies we get

$$H^q(\mathfrak{g}, K; V) = H^q(\mathfrak{g}, \mathfrak{k}; V)^{\pi_0(G)},$$

as taking invariants of a semisimple group action is plainly exact.

3.3 de Rham isomorphism

Now suppose that V^{smooth} is a smooth admissible representation of G , which contains V as K -finite vectors. Denote by $\Omega^q(G/K; V^{\text{smooth}})$ the space of smooth V^{smooth} -valued differential q -forms, i.e. the space of smooth sections $\omega : G/K \rightarrow \bigwedge^q T^*(G/K) \otimes V^{\text{smooth}}$. Then $\Omega^q(G/K; V^{\text{smooth}})$ becomes a complex with the natural exterior differential. Furthermore we have a natural action of G given by

$$(g \cdot \omega)_x(X) := g \cdot \omega_{g^{-1}x}(g^{-1}X),$$

for any $x \in G/K$ and any $X \in \bigwedge^q T_x(G/K)$.

As G acts transitively on G/K , we have a natural isomorphism

$$\Omega^q(G/K; V^{\text{smooth}})^G \cong C^q(\mathfrak{g}, K; V),$$

given by evaluation

$$\omega \mapsto \omega_e,$$

by virtue of the identification $T_e(G/K) = \mathfrak{p}$. Indeed, due to the admissibility the canonical inclusion induces for the K -invariant origin $e \in G/K$ an isomorphism of stalks

$$\left(\bigwedge^q T_e^*(G/K) \otimes V \right)^K \cong \left(\bigwedge^q T_e^*(G/K) \otimes V^{\text{smooth}} \right)^K.$$

As the differentials of these two complexes are compatible, we get a natural isomorphism

$$H^q(\Omega^\bullet(G/K; V^{\text{smooth}})^G) \cong H^q(\mathfrak{g}, K; V)$$

in cohomology. A similar statement holds for $(\mathfrak{g}, \mathfrak{k})$ -cohomology. The same reasoning yields canonical isomorphisms

$$H^q(\Omega^\bullet(G/K; V^{\text{smooth}})) \cong H^q(\Omega^\bullet(G/K; V^{\text{smooth}})^{G^0}) \cong H^q(\mathfrak{g}, \mathfrak{k}; V),$$

the first isomorphism being classically due to de Rham.

3.4 Component action

We have seen that $H^q(\mathfrak{g}, K; V)$ is the subspace of $H^q(\mathfrak{g}, \mathfrak{k}; V)$ where $\pi_0(K) = \pi_0(G)$ acts trivially. In our applications it turns out that we also need to consider nontrivial eigen spaces of this action.

Let ε be a character of $\pi_0(K)$, that we consider also as a character of K via the projection $K \rightarrow \pi_0(K)$. Then we have a corresponding eigen space

$$H^q(\mathfrak{g}, \mathfrak{k}; V)_\varepsilon := \{h \in H^q(\mathfrak{g}, \mathfrak{k}; V) \mid \forall k_0 \in \pi_0(K) : k_0 h = \varepsilon(k_0) \cdot h\}.$$

Obviously for the trivial character we get

$$H^q(\mathfrak{g}, \mathfrak{k}; V)_1 = H^q(\mathfrak{g}, K; V).$$

We consider ε as the one-dimensional (\mathfrak{g}, K) -module, on which (\mathfrak{g}, K^0) acts trivially and on which $\pi_0(K)$ acts via ε . This corresponds to the pullback of ε along $G \rightarrow \pi_0(K)$.

In our application V comes from the infinity component of an irreducible cuspidal automorphic representation and furthermore has non-trivial (\mathfrak{g}, K^0) -cohomology. Then two cases will arise. The first one concerns even n . In this case V will be isomorphic to $V \otimes \varepsilon$ for all ε and we have non-vanishing eigen spaces in cohomology, all of the same dimensions for any ε . For odd n this is not the case and there is a unique choice of ε such that the eigen space in cohomology does *not* vanish.

3.5 Cohomology of arithmetic groups

We keep the above notation and let $\rho : G \rightarrow E$ denote a smooth finite-dimensional representation of G . We can consider the smooth G -module $V^{\text{smooth}} \otimes E$. Fix an arithmetic subgroup Γ of G contained in G^0 . Then ρ induces a finite dimensional representation of Γ , that we also denote E . It is classical that we have canonically

$$H^q(\Gamma; E) \cong H^q(\Omega^\bullet(G/K; E)^\Gamma).$$

Assume for simplicity that Γ acts freely. The general case can be deduced from this case via the Hochschild-Serre spectral sequence, as this implies in particular that for a (torsion-free) normal subgroup $\Gamma' \subseteq \Gamma$ of finite index there is a natural isomorphism

$$H^q(\Gamma; E) = H^q(\Gamma'; E)^{\Gamma/\Gamma'},$$

and we have a similar statement for the complex of smooth E -valued differential forms.

We know that G/K is an Euclidean space, in particular it is contractible. As Γ is torsion-free, it acts freely on this space and $\Gamma \backslash G/K$ is an Eilenberg-MacLane space $K(\Gamma, 1)$. It results that

$$H^q(\Gamma; E) \cong H^q(\Gamma \backslash G/K; \underline{E}),$$

where \underline{E} is the local system associated to E , that we consider as a sheaf on $\Gamma \backslash G/K$. For any open $U \subseteq \Gamma \backslash G/K$ the sections are given by locally constant functions that are invariant under Γ , i.e.

$$\underline{E}(U) = \{\phi : \Gamma U \rightarrow E \mid \forall x \in \Gamma U : \phi(x) = \rho(\gamma)(\phi(\gamma^{-1}x))\},$$

where ΓU denotes the preimage of U under the canonical projection $\pi : G/K \rightarrow \Gamma \backslash G/K$. Note that this sheaf cohomology can be calculated by our de Rham complex, as the restriction maps are locally constant. For the latter the pullback

$$\pi^* : \Omega^q(\Gamma \backslash G/K; \underline{E}) \rightarrow \Omega^q(G/K; E),$$

$$\omega \mapsto \omega \circ \pi$$

along π , where on the right hand side we consider E as a constant sheaf, induces an isomorphism

$$\Omega^q(\Gamma \backslash G/K; \underline{E}) \cong \Omega^q(G/K; E)^\Gamma.$$

We conclude that

$$H^q(\Gamma; E) \cong H^q(\Omega^\bullet(G/K; E)^\Gamma).$$

On the other hand translation by $g^{-1} \in G$ yields a natural identification of tangent spaces

$$T_g(G) \rightarrow T_e(G) = \mathfrak{g}.$$

Therefore we can canonically identify

$$\Omega^q(G; E) = \text{Hom}\left(\bigwedge^q \mathfrak{g}, \mathcal{C}^\infty(G; E)\right).$$

Now Γ acts on these spaces in a compatible manner by virtue of a trivial action on the tangent spaces. Then

$$\omega \in \Omega^q(G; E)^\Gamma$$

if and only if for any $\gamma \in \Gamma$, $x \in G$, $X \in \bigwedge^q T_x(G) = \mathfrak{g}$,

$$\omega_x(X) = \rho(\gamma)(\omega_{\gamma^{-1}x}(X)).$$

In particular the above identification yields a canonical isomorphism

$$\Omega^q(G; E)^\Gamma = \text{Hom}\left(\bigwedge^q \mathfrak{g}, \mathcal{I}_\Gamma^\infty(G; E)\right),$$

where

$$\mathcal{I}_\Gamma^\infty(G; E) := \{\phi \in \mathcal{C}^\infty(G; E) \mid \forall \gamma \in \Gamma, x \in G : \phi(x) = \rho(\gamma)(\phi(\gamma^{-1}x))\}.$$

On this space G acts by right translation, i.e. this gives the representation smoothly induced from the restriction of ρ to Γ . Our identifications are compatible with the differentials of our complexes, such that we get

$$H^q(\Omega^\bullet(G; E)^\Gamma) \cong H^q(\mathfrak{g}; \mathcal{I}_\Gamma^\infty(G; E)),$$

whence our first step towards identifying the cohomology of the arithmetic group with relative Lie algebra cohomology. Now the map

$$\phi \mapsto \phi^0 : g \mapsto \rho(g)^{-1}(\phi(g))$$

gives an identification

$$\mathcal{I}_\Gamma^\infty(G, E) \cong \mathcal{C}^\infty(\Gamma \backslash G; E) = \mathcal{C}^\infty(\Gamma \backslash G) \otimes E,$$

of G -modules, where G acts on the right hand side by right translation and ρ respectively. So we get an isomorphism

$$H^q(\Gamma \backslash G; \underline{E}) \cong H^q(\Omega^\bullet(G; E)^\Gamma) \cong H^q(\mathfrak{g}; \mathcal{I}_\Gamma^\infty(G; E)) \cong H^q(\mathfrak{g}; \mathcal{C}^\infty(\Gamma \backslash G) \otimes E).$$

Eventually the same procedure works for symmetric spaces and relative Lie algebra cohomology. To be more precise, write $\pi : G \rightarrow G/K$ for the canonical projection. Then pullback along π induces an isomorphism of complexes

$$\pi^* : \Omega^q(G/K; E)^\Gamma \cong C^q(\mathfrak{g}, K; \mathcal{I}_\Gamma^\infty(G; E)) \cong C^q(\mathfrak{g}, \mathfrak{k}; \mathcal{I}_\Gamma^\infty(G^0; E)),$$

which in turn induces an isomorphism in cohomology:

$$H^q(\Gamma; E) \cong H^q(\Omega^\bullet(G/K; E)^\Gamma) \cong H^q(\mathfrak{g}, \mathfrak{k}; \mathcal{I}_\Gamma^\infty(G^0; E)) \cong$$

$$H^q(\mathfrak{g}, \mathfrak{k}; \mathcal{C}^\infty(\Gamma \backslash G^0) \otimes E) \cong H^q(\mathfrak{g}, K; \mathcal{C}^\infty(\Gamma \backslash G) \otimes E).$$

This construction can be exploited mutatis mutandis with growths conditions and compact support, i.e. we always have an isomorphism

$$H_*^q(\Gamma \backslash G/K; \underline{E}) \cong H_*^q(\mathfrak{g}, K; \mathcal{C}_*^\infty(\Gamma \backslash G) \otimes E),$$

where $*$ \in $\{c, \text{cusp}, \text{fd}, \text{mg}\}$. The beauty of this isomorphism is that the right hand side can be computed purely algebraically, because it does not change when restricting to K -finite vectors. Finally we note that once a complex structure on the symmetric space is available, this result can be refined and shown to naturally respect the respective Hodge decompositions.

In suitable situations it can even be guaranteed that

$$H^q(\mathfrak{g}, K^0; V \otimes E) = \left(\bigwedge^q \mathfrak{p}^* \otimes V \otimes E \right)^{K^0},$$

which is the case when V^{smooth} is unitary and E an algebraic representation. The reason being that, as is well known, the Casimir operators of the representations act by scalars, so we might apply [BW80, Chapter II, Proposition 3.1] to the connected component, which yields the result for $(\mathfrak{g}, \mathfrak{k})$ -cohomology. Taking K -invariants we might recover (\mathfrak{g}, K) -cohomology. This is a vast generalization of a classical result of E. Cartan saying that on G/K all G^0 -invariant forms are harmonic, closed and co-closed.

We pick a closed connected θ -stable subgroup S in the center of G , with Lie algebra \mathfrak{s} . Then we consider the manifold G/KS whose tangent space at the identity is given by

$$\mathfrak{g}/(\mathfrak{k} + \mathfrak{s}) = \mathfrak{p}/(\mathfrak{p} \cap \mathfrak{s}).$$

Obviously we might assume $\mathfrak{s} \subseteq \mathfrak{p}$, which is the same as to say that all compact subgroups in S are trivial. We assume that $S \cap \Gamma$ is trivial. This is in particular the case if S is the connected component of the \mathbf{R} -valued points of a \mathbf{Q} -split torus in the center of G_n . Then all of the above results hold with the following modifications. We define the (\mathfrak{g}, KS) -cohomology as the cohomology of the complex

$$C^q(\mathfrak{g}, K; V)^S = C^q(\mathfrak{g}, \mathfrak{k} + \mathfrak{s}; V)^{\pi_0(G)},$$

for which we write $H^q(\mathfrak{g}, KS; V)$. Then

$$H^q(\Gamma; E) \cong H^q(\Omega^\bullet(G/KS; E)^\Gamma) \cong H^q(\mathfrak{g}, K; \mathcal{I}_\Gamma^\infty(G/S; E)) \cong$$

$$H^q(\mathfrak{g}/\mathfrak{s}, KS; \mathcal{C}^\infty(\Gamma \backslash G/S) \otimes E),$$

by the very same argument.

4 Cohomological construction of the distribution

In this section we suppose that k is totally real and that π and σ are *regular* and *algebraic* in the sense of [Clo90, Definition 1.8, Definition 3.12]. We use the modular symbols constructed in [Jan11] with possibly nontrivial coefficients to deduce that the distribution above is in fact algebraic and \mathfrak{p} -adically bounded, i.e. a \mathfrak{p} -adic measure. For the mere treatment of algebraicity over \mathbf{Q} see [KS09] and for the case of trivial coefficients over an arbitrary number field consult [Jan11].

4.1 Cohomological interpretation of the period integrals

We identify S_∞ with the set of embeddings $k \rightarrow \mathbf{R}$. We fix the standard torus T_n in GL_n and consider all root data for GL_n resp. G_n with respect to T_n resp. $\mathrm{Res}_{k/\mathbf{Q}} T_n$ and the ordering induced by the standard Borel subgroup B_n resp. $\mathrm{Res}_{k/\mathbf{Q}} B_n$. We make the usual standard choice of simple roots, and we choose as basis of characters the component projections $\chi_j : T_n \rightarrow \mathbf{G}_m$, $(t_i) \mapsto t_j$. Furthermore for any $\iota \in S_\infty$ we are given an irreducible representation ρ_{μ_ι} of GL_n of highest weight $\mu_\iota = (\mu_{\iota,i})_{1 \leq i \leq n} \in \mathbf{Z}^n = X(T_n)$, which is supposed to be dominant and regular. Likewise we have an irreducible representation of highest weight ν_ι of GL_{n-1} . For $\mu = (\mu_\iota)_{\iota \in S_\infty}$ we write M_μ for a fixed \mathbf{Z} -model of the representation space of $\rho_\mu = \otimes_{\iota \in S_\infty} \rho_{\mu_\iota}$, i.e. a \mathbf{Z} -scheme in modules. In particular we can consider for any algebra A its A -valued points $M_{\mu_\iota}(A) = M_{\mu_\iota}(\mathbf{Z}) \otimes_{\mathbf{Z}} A$. Then we consider M_μ to be the corresponding irreducible representation of $G_n = \mathrm{Res}_{\mathcal{O}_k/\mathbf{Z}} \mathrm{GL}_n$, the latter canonically identified

with $\mathrm{GL}_n^{S_\infty}$ after sufficient extension of scalars. We have a well defined diagonal embedding $\mathrm{GL}_n \rightarrow G_n$ over \mathbf{Z} . In particular we write w_n for the long Weyl element in GL_n , embedded diagonally into G_n . It should be clear from the context when we are talking about a diagonally embedded w_n or not.

To clarify the relation between finite dimensional representations and the critical values we need to introduce some more notation and terminology. See [Jan11, section 1] for the details concerning the general setup for reductive groups. Write $G_n := \mathrm{Res}_{\mathcal{O}_k/\mathbf{Z}} \mathrm{GL}_n$ and introduce the group

$${}^0G_n := \bigcap_{\alpha \in X_{\mathbf{Q}}(G)} \ker \alpha^2.$$

Then 0G_n is a reductive group scheme over \mathbf{Z} and $X_{\mathbf{Q}}({}^0G_n) \otimes_{\mathbf{Z}} \mathbf{Q} = 1$ [Jan11, Proposition 1.2]. Furthermore denote by S the maximal \mathbf{Q} -split torus in the radical of G_n , or in the maximal \mathbf{Q} -split central torus of G_n , what amounts to the same. Then

$$G_n(\mathbf{R}) = {}^0G_n(\mathbf{R}) \rtimes S(\mathbf{R})^0,$$

cf. [BS73, Proposition 1.2]. In our case $S(\mathbf{R})^0$ is isomorphic to $\mathbf{G}_m(\mathbf{R})^0$ as a (real) Lie group and furthermore

$$\mathrm{rank}_{\mathbf{R}} \mathcal{C}(G_n) = [k : \mathbf{Q}]$$

as k is totally real, cf. [Jan11, section 5]. We have explicitly

$${}^0G_n(\mathbf{R}) = \{g \in G_n(\mathbf{R}) \mid \prod_{\iota \in S_\infty} \|\det g_\iota\|_\iota = 1\},$$

furthermore we introduce the subgroup

$$G_n^\pm = \{g \in G_n(\mathbf{R}) \mid \forall \iota \in S_\infty : \|\det g_\iota\|_\iota = 1\}.$$

The reason for considering the latter lies in the fact that, contrary to the former, it has a natural decomposition into local components, which allows to apply the Künneth formalism to study its Lie algebra cohomology, which in turn is related to the de Rham cohomology of the symmetric spaces associated to the former or equivalently to the Lie algebra cohomology of that group.

We suppose that for all $\iota \in S_\infty$

$$H^\bullet(\mathfrak{gl}_{n,\iota}, K_{n,\iota}; \pi_\iota \otimes M_{\boldsymbol{\mu}_\iota}(\mathbf{C})) \neq 0,$$

and likewise

$$H^\bullet(\mathfrak{gl}_{n-1,\iota}, K_{n-1,\iota}; \sigma_\iota \otimes M_{\boldsymbol{\nu}_\iota}(\mathbf{C})) \neq 0,$$

where $\mathfrak{gl}_{n,\iota}$ is the komplexified Lie algebra of $\mathrm{GL}_n(k_\iota)$ and $K_{n,\iota} := \iota(O(n)Z_n^0)$ is the product of the connected component of the center of $\mathrm{GL}_n(k_\iota)$ and its standard maximal compact subgroup. Now regularity of π means that the $\boldsymbol{\mu}_{\iota,i}$ are pairwise distinct, i.e. $\boldsymbol{\mu}_{\iota,i} > \boldsymbol{\mu}_{\iota,i+1}$ for all $1 \leq i < n$. Furthermore we know that the *weight*

$$\mathbf{w} := \boldsymbol{\mu}_{\iota,i} + \boldsymbol{\mu}_{\iota,n+1-i}$$

is *independent* of i and $\iota \in S_\infty$, and similarly

$$\mathbf{v} := \boldsymbol{\nu}_{\iota, i} + \boldsymbol{\nu}_{\iota, n-i}.$$

In other words, π_∞ and σ_∞ are selfcontragredient up to twist.

We write for any $\mu \in \mathbf{Z}$

$$M_\mu[\mu] := M_\nu \otimes M_{(\mu)},$$

where $M_{(\mu)}$ denotes the one-dimensional G_n -module on which G_n acts via the projection $G_n \rightarrow G_n / ({}^0G_n)^0 \cong \mathbf{G}_m$, and \mathbf{G}_m acts via the character $x \mapsto x^\mu$. Assume that our integral models are chosen in such a way that we can fix for all dominant weights $\boldsymbol{\mu}$ and all $\mu \in \mathbf{Z}$ isomorphisms

$$M_\nu[\nu] \rightarrow M_{\nu+\nu}, \quad (5)$$

where $\boldsymbol{\mu} + \mu$ is considered to be the collection of weights $(\boldsymbol{\mu}_{\iota, i} + \mu)_{1 \leq i \leq n}$.

We write $\check{\boldsymbol{\mu}}$ for the highest weight of the contragredient representation corresponding to $\boldsymbol{\mu}$, and assume that we have another twisted isomorphism

$$\cdot^\vee : M_\mu \rightarrow M_{\check{\boldsymbol{\mu}}},$$

which induces an isomorphism if we twist the action on M_μ by the twisted main involution $g \mapsto w_n(g^{-1})^t w_n$, for the long Weyl element $w_n \in \text{GL}_n$. Furthermore we assume that \cdot^\vee is compatible with the above isomorphisms.

By a straightforward computation (cf. [KS09, Proposition 2.2]) we know that there exists a critical half-integer $s = \frac{1}{2} + \nu$ for $L(s, \pi \times \sigma)$ if and only if

$$\mathbf{w} \equiv \mathbf{v} \pmod{2} \quad (6)$$

and in that case all critical half-integers are centered around

$$\frac{1 + \mathbf{w} + \mathbf{v}}{2},$$

which is itself critical. Furthermore this condition of criticality is equivalent to the existence of a G_{n-1} -equivariant embedding $M_\nu[\nu] \rightarrow M_\mu$. Then the map

$$\nu \mapsto s = \frac{1}{2} + \nu$$

sets up a bijection between the set $\text{Emb}(\check{\boldsymbol{\nu}}, \boldsymbol{\mu})$ of integers ν such that there is an embedding $M_\nu[\nu] \rightarrow M_\mu$ of G_{n-1} -modules and the set of critical half-integers for $L(s, \pi \times \sigma)$.

Denote (\mathbf{w}, \mathbf{l}) the Langlands parameter for π_∞ , i.e.

$$\mathbf{l} = 2 \cdot (\boldsymbol{\mu} + \rho_n) - (\mathbf{w}),$$

where ρ_n denotes the half sum of the positive roots for our choice of root datum in G_n and (\mathbf{w}) denotes the diagonally embedded weight \mathbf{w} . Then

$\mathbf{l} = (l_{\iota,i})_{\iota \in S_\infty, 1 \leq i \leq n}$ and we use a similar notation for the Langlands parameter (\mathbf{v}, \mathbf{m}) for σ_∞ . With this notation we set

$$\nu_{\min} := \frac{\mathbf{w} + \mathbf{v}}{2} - \min_{i,j,\iota} |l_{\iota,i} - m_{\iota,j}| + 1.$$

Note that due to our parity condition (6), we know that

$$\nu_{\min} \leq \frac{\mathbf{w} + \mathbf{v}}{2}.$$

With this notation

$$s_{\min} := \frac{1}{2} + \nu_{\min}$$

is the left most critical value for $L(s, \pi \otimes \sigma)$, and the right most is

$$s_{\max} := \frac{1}{2} + \mathbf{w} + \mathbf{v} - \nu_{\min}.$$

Write $\mathcal{X}_n, {}^0\mathcal{X}_n, \mathcal{X}_n^{\text{ad}}$ for the symmetric spaces associated to $G_n(\mathbf{R}), {}^0G_n(\mathbf{R})$ and $G_n^{\text{ad}}(\mathbf{R})$ respectively. We use the same superscripts for arithmetic subgroups and their various projections and restrictions. Note however that in this sense ${}^0\Gamma = \Gamma$ for any arithmetic $\Gamma \subseteq G_n(\mathbf{R})$, so that we drop the superscript ‘0’ from the notation. Considering arithmetic quotients we also tend to drop the superscript ‘ad’.

Consider the composition

$$s := \text{ad} \circ j : G_{n-1} \rightarrow G_n \rightarrow G_n^{\text{ad}}$$

of group schemes over \mathbf{Z} . This is eventually a central morphism which can be interpreted in cohomology with suitable growths conditions by means of [Jan11, Proposition 1.4]. For any $\gamma \in G_n^{\text{ad}}(\mathbf{Q})$ and sufficiently compatible congruence subgroups $\Gamma_{n-1} \subseteq G_{n-1}(\mathbf{R}), \Gamma_n^{\text{ad}} \subseteq G_n^{\text{ad}}(\mathbf{Z})$, i.e. $\gamma^{-1}s(\Gamma_{n-1})\gamma \subseteq \Gamma_n^{\text{ad}}$, we have a natural map in cohomology

$$s_\gamma^* : H_c^q(\Gamma_n^{\text{ad}} \backslash \mathcal{X}_n^{\text{ad}}) \rightarrow H_c^q(\Gamma_{n-1} \backslash \mathcal{X}_{n-1}),$$

which is induced by the pullback s_γ^* along the map

$$\begin{aligned} s_\gamma : \Gamma_{n-1} \backslash \mathcal{X}_{n-1} &\rightarrow \Gamma_n^{\text{ad}} \backslash \mathcal{X}_n^{\text{ad}}, \\ \Gamma_{n-1}x &\mapsto \Gamma_n^{\text{ad}}\gamma^{-1}s(x), \end{aligned}$$

which is proper in fact.

Now when considering M_μ as a GL_{n-1} -module that we denote $j^*(M_\mu)$, it obviously decomposes into a finite sum of irreducible modules

$$j^*(M_\mu) = \bigoplus_{\mu^*} M_{\mu^*},$$

where μ^* is a highest weight for G_{n-1} . By the classical work of Weyl [Wey39] the sum ranges over all such μ^* satisfying for all $\iota \in S_\infty$ and all $1 \leq i < n$

$$\mu_{\iota,i} \geq \mu_{\iota,i}^* \geq \mu_{\iota,i+1}.$$

We claim that

$$\mathrm{Hom}_{\Gamma_{n-1}}(M_{\check{\nu}}, j^*(M_{\mu}))$$

has a natural basis enumerated by $\mathrm{Emb}(\check{\nu}, \mu)$.

Indeed, Borel has shown that $\Gamma_{n-1}^{\mathrm{der}} := \Gamma_{n-1} \cap G_{n-1}^{\mathrm{der}}(\mathbf{R})$ is Zariski dense in G_{n-1}^{der} . As we know that $\Gamma_{n-1} \backslash {}^0\mathcal{X}_{n-1}$ is of finite invariant volume and that there is a split maximal torus in ${}^0G_{n-1}$ over \mathbf{R} , the argument of the proof of [Jan11, Lemma 1.6] shows that Γ_{n-1} is eventually Zariski dense in $({}^0G_{n-1})^0$ (this statement is false if k has a complex place, as the classical Theorem of Dirichlet on the structure of the integral units reveals). Furthermore it is easily seen that $G_{n-1} = ({}^0G_{n-1})^0 S$. We deduce that there exists a Γ_{n-1} -linear isomorphism

$$M_{\check{\nu}} \rightarrow M_{\mu^*}$$

if and only if for some $\nu \in \mathbf{Z}$ we have the identity of weights

$$\check{\nu} + \nu = \mu^*.$$

For any ν we have a natural isomorphism of G_{n-1} -modules

$$M_{\nu} \otimes M_{\check{\nu}+\nu} \rightarrow M_{(\nu)},$$

where the latter denotes the scheme in free modules of rank 1, on which G_{n-1} acts via its projection to the torus $G_{n-1}/({}^0G_{n-1})^0$, and the latter acts by the character $s \mapsto s^\nu$. In particular we have natural isomorphisms

$$H^0(\Gamma_{n-1}; j^*(M_{\mu}) \otimes M_{\nu}) \cong H^0({}^0G_{n-1})^0; j^*(M_{\mu}) \otimes M_{\nu})$$

and

$$\tau : H^0(\Gamma_{n-1}; j^*(M_{\mu}) \otimes M_{\nu}) \rightarrow \bigoplus_{\nu \in \mathrm{Emb}(\check{\nu}, \mu)} M_{(\nu)} \quad (7)$$

of G_{n-1} -modules and likewise a natural equivariant projection

$$\tau_{\nu} : H^0(\Gamma_{n-1}; j^*(M_{\mu}) \otimes M_{\nu}) \rightarrow M_{(\nu)}.$$

Write $\underline{M}_{\nu}(A)$ for the sheaf associated to the module $M_{\nu}(A)$ on $\Gamma_{n-1} \backslash \mathcal{X}_{n-1}$. As $\Gamma_{n-1} \backslash \mathcal{X}_{n-1}$ is orientable, we can fix an orientation, which means that we fix an isomorphism $H_c^{d_{n-1}}(\Gamma_{n-1} \backslash \mathcal{X}_{n-1}, \underline{A}) \cong A$. Assume that A is a noetherian and integrally closed domain. Then any free A -module of finite rank is reflexive. We have a natural perfect pairing

$$\lambda : (M_{\mu}(A) \otimes_A M_{\nu}(A)) \otimes_A (M_{\mu}(A) \otimes_A M_{\nu}(A))^{\vee} \rightarrow A.$$

This pairing identifies the right argument as the A -dual of the left argument, including the canonical $G_{n-1}(A)$ -actions.

Plugging things together we get for any $p \in \mathbf{Z}$ a natural pairing

$$\begin{aligned} H_c^p(\Gamma_{n-1} \setminus \mathcal{X}_{n-1}; \underline{M}_\mu(A) \otimes_A \underline{M}_\nu(A)) \otimes_A H^{d_{n-1}-p}(\Gamma_{n-1} \setminus \mathcal{X}_{n-1}^{\text{ad}}; (\underline{M}_\mu(A) \otimes_A \underline{M}_\nu(A))^\vee) \\ \rightarrow H_c^{d_{n-1}}(\Gamma_{n-1} \setminus \mathcal{X}_{n-1}; \underline{A}) \rightarrow A, \end{aligned}$$

by composing $H_c^{d_{n-1}}(\lambda)$ with the \cup -product. By Poincaré, this is a perfect pairing. In particular we might identify

$$H_c^{d_{n-1}}(\Gamma_{n-1} \setminus \mathcal{X}_{n-1}; s_\gamma^*(\underline{M}_\mu(A)) \otimes_A \underline{M}_\nu(A))$$

with the A -dual of

$$H^0(\Gamma_{n-1} \setminus \mathcal{X}_{n-1}^{\text{ad}}; (s_\gamma^*(\underline{M}_\mu(A)) \otimes_A \underline{M}_\nu(A))^\vee),$$

which turns out to be¹

$$H^0(\Gamma_{n-1} \setminus \mathcal{X}_{n-1}^{\text{ad}}; s_\gamma^*(\underline{M}_\mu(A)) \otimes_A \underline{M}_\nu(A)),$$

taking the $G_{n-1}(A)$ -action into account. Hence we get a canonical natural isomorphism

$$\int_{\Gamma_{n-1} \setminus \mathcal{X}_{n-1}} : H_c^{d_{n-1}}(\Gamma_{n-1} \setminus \mathcal{X}_{n-1}; s_\gamma^*(\underline{M}_\mu(A)) \otimes_A \underline{M}_\nu(A)) \rightarrow \quad (8)$$

$$H^0(\Gamma_{n-1} \setminus \mathcal{X}_{n-1}; s_\gamma^*(\underline{M}_\mu(A)) \otimes_A \underline{M}_\nu(A)).$$

Consequently we get by Poincaré duality a natural map

$$\mathcal{B}_\gamma^{\Gamma_n, \Gamma_{n-1}} : H_c^q(\Gamma_n \setminus \mathcal{X}_n^{\text{ad}}; \underline{M}_\mu(A)) \otimes_A H_c^{d_{n-1}-q}(\Gamma_{n-1} \setminus \mathcal{X}_{n-1}^{\text{ad}}; \underline{M}_\nu(A)) \rightarrow$$

$$H^0(\Gamma_{n-1} \setminus \mathcal{X}_{n-1}; s_\gamma^*(\underline{M}_\mu) \otimes \underline{M}_\nu(A)),$$

which for $A = \mathbf{C}$ is given by

$$\alpha \otimes \beta \mapsto \int_{\Gamma_{n-1} \setminus \mathcal{X}_{n-1}} s_\gamma^*(\alpha) \wedge \text{ad}^*(\beta),$$

where $d_{n-1} = \dim \mathcal{X}_{n-1}$.

Note that the map

$${}^0\text{ad} : \Gamma_{n-1} \setminus {}^0\mathcal{X}_{n-1} \rightarrow \Gamma_{n-1} \setminus \mathcal{X}_{n-1}^{\text{ad}}$$

induced by ad is proper [Jan11, Lemma 1.5], and the domain is of finite invariant volume. Furthermore integration along the fibre of the natural decomposition

$$G_{n-1}(\mathbf{R}) = {}^0G_{n-1}(\mathbf{R}) \rtimes S(\mathbf{R})^0,$$

¹reflexivity...

induces a natural map

$$i_* \circ s_\gamma^* : H_{\text{fd}}^q(\Gamma_n \setminus \mathcal{X}_n^{\text{ad}}) \rightarrow H_{\text{mg}}^{q-1}(\Gamma_{n-1} \setminus \mathcal{X}_{n-1}).$$

Consequently we may identify the above topological symbol with the relative modular symbols constructed in [KMS00, Jan11]. Our definition of the modular symbol is essentially a generalization of that of [Sch01].

We might compose with τ_ν to identify the image of $\mathcal{B}_\gamma^{\Gamma_n, \Gamma_{n-1}}$ as the space $\mathbf{C}^{\text{Emb}(\check{\nu}, \boldsymbol{\mu})}$, because observe that

$$H^0(\Gamma_{n-1} \setminus \mathcal{X}_{n-1}; (j^*(\underline{M}_\mu) \otimes \underline{M}_\nu)(\mathbf{C})) = H^0(\Gamma_{n-1}; (j^*(M_\mu) \otimes M_\nu)(\mathbf{C})),$$

by means of the definition of the sheaves, and conjugation by γ induces an automorphism of $M_\mu(\mathbf{C})$, such that we have an isomorphism

$$s_\gamma^*(\underline{M}_\mu(\mathbf{C})) \cong s^*(\underline{M}_\mu(\mathbf{C})),$$

and the latter may be canonically identified with a subsheaf of $j^*(\underline{M}_\mu(\mathbf{C}))$. Note however that this is no longer true when we take integral structures into account. Nonetheless the components have the following interpretation.

Pick some compactly supported (or some fast decreasing) differential forms

$$\alpha^0 = \omega \otimes \phi \in \left(\bigwedge^q (\mathfrak{g}_n / (\mathfrak{k}_n + \mathfrak{s}))^* \otimes \mathcal{C}^\infty(\Gamma_n \setminus G_n^{\text{ad}}(\mathbf{R}); M_\mu(\mathbf{C})) \right)^{K_n^0}.$$

We already sketched the construction of how to interpret this as a classical differential form. In particular we set for any $g \in G_n^{\text{der}}(\mathbf{R})$

$$\alpha := \omega \otimes (g \mapsto \rho_\mu(g)(\phi(\text{ad}(g)))),$$

then via the identification $\mathcal{X}_n^{\text{ad}} = \mathcal{X}_n^{\text{der}}$

$$\alpha \in \Omega^q(\mathcal{X}_n^{\text{ad}}; M_\mu(\mathbf{C}))^{\Gamma_n}.$$

We might pushforward this form along the canonical projection to the arithmetic quotient and get a form

$$\alpha \in \Omega^q(\Gamma_n \setminus \mathcal{X}_n^{\text{ad}}; \underline{M}_\mu(\mathbf{C})).$$

We apply the very same procedure to an element

$$\beta^0 = \omega' \otimes \varphi \in \left(\bigwedge^{d_n - q} (\mathfrak{g}_{n-1} / (\mathfrak{k}_{n-1} + \mathfrak{s}))^* \otimes \mathcal{C}^\infty(\Gamma_{n-1} \setminus G_{n-1}^{\text{ad}}(\mathbf{R}); M_\nu(\mathbf{C})) \right)^{K_{n-1}^0}.$$

Likewise we get a form

$$\beta \in \Omega^{d_n - q}(\Gamma_{n-1} \setminus \mathcal{X}_{n-1}^{\text{ad}}; \underline{M}_\nu(\mathbf{C})).$$

The pullback $\overline{\varphi} : \Gamma_{n-1} \backslash \mathcal{X}_{n-1} \rightarrow M_{\nu}(\mathbf{C})$ of φ along $\Gamma_{n-1} \backslash \mathcal{X}_{n-1} \rightarrow \Gamma_{n-1} \backslash \mathcal{X}_{n-1}^{\text{ad}}$ is given explicitly by

$$gz \mapsto \rho_{\nu}(z)^{-1} \varphi(g),$$

where $z \in \mathcal{C}(G_{n-1})(\mathbf{R})^0$. We apply the same notation to the pullback $\overline{\phi} : \Gamma_n \backslash \mathcal{X}_n \rightarrow M_{\mu}(\mathbf{C})$ along the analogous map. Then by the isomorphism (7) we see that the above pairing reads

$$\alpha \otimes \beta \mapsto \left(\tau \circ \int_{\Gamma_{n-1} \backslash \mathcal{X}_{n-1}} \tau_{\nu}(\rho_{\nu+\nu}(g) \otimes \rho_{\nu}(g)) (\overline{\phi}(s_{\gamma}(g)) \otimes \overline{\varphi}(g)) dg \right)_{\nu \in \text{Emb}(\check{\nu}, \mu)}$$

where the right hand side is an element of $\mathbf{C}^{\text{Emb}(\nu, \check{\mu})}$. The components of the right hand side read

$$\begin{aligned} \int_{\Gamma_{n-1} \backslash \mathcal{X}_{n-1}} \int_{S(\mathbf{R})^0} \tau_{\nu}(\rho_{\nu+\nu}(gs) \otimes \rho_{\nu}(gs)) (\overline{\phi}(s_{\gamma}(gs)) \otimes \overline{\varphi}(gs)) ds dg = \\ \int_{\Gamma_{n-1} \backslash \mathcal{X}_{n-1}} \overline{\phi}(s_{\gamma}(g)) \overline{\varphi}(g) s(g)^{\nu} dg, \end{aligned}$$

where $s(g)$ denotes the projection of g to the split torus $G_{n-1}/({}^0G_{n-1})^0 \cong \mathbf{G}_m$ and the components of the first two formulas are identified with their canonical projections to the $({}^0G_{n-1})^0$ -invariants.

4.2 Construction of cohomology classes

If we make in particular the following choice for ϕ and φ , we eventually recover the period integrals of the previous section. Assume that we have non-vanishing cohomologies

$$\begin{aligned} H^{\bullet}(\tilde{\mathfrak{g}}_n, K_n; \pi_{\infty} \otimes M_{\mu}(\mathbf{C})) &\neq 0, \\ H^{\bullet}(\tilde{\mathfrak{g}}_{n-1}, K_{n-1}; \sigma_{\infty} \otimes M_{\nu}(\mathbf{C})) &\neq 0, \end{aligned}$$

where $\tilde{\mathfrak{g}}_n$ denotes the complexified Lie algebra of $G_n^{\text{ad}}(\mathbf{R})$ or equivalently of $(G_n^{\pm})^0$, and K_n is the product of a maximal compact subgroup and the center of $G_n(\mathbf{R})$. We tacitly assumed the infinity components to be smooth. Note that this condition of cohomologicality is only slightly stricter than allowing quadratic twists. This assumption simplifies our formulation.

Now for the unique even integer $m \in \{n, n-1\}$ we are eventually interested in the eigen spaces

$$H^{r b_m}(\tilde{\mathfrak{g}}_m, K_m^0; \tau_{\infty} \otimes M_{\eta}(\mathbf{C}))_{\varepsilon} \cong \mathbf{C},$$

where η is μ if $m = n$ or ν otherwise, likewise τ_{∞} is π_{∞} or σ_{∞} , and ε is a character of $\pi_0(G_m(\mathbf{R}))$. These eigen spaces are known to be one-dimensional, where the bottom degree is given by $b_m = \frac{n^2 - n + 2 \lfloor \frac{n}{2} \rfloor}{2}$, $r = [k : \mathbf{Q}]$.

Write $m' \in \{n, n-1\} \setminus \{m\}$ for the unique odd rank of interest, $\boldsymbol{\eta}'$ for the corresponding weight and τ'_∞ for the representation at infinity respectively. In this case

$$H^{rb_{m'}}(\tilde{\mathfrak{g}}_{m'}, K'_m; \tau'_\infty \otimes M_{\boldsymbol{\eta}'}(\mathbf{C})) \cong \mathbf{C}$$

is one-dimensional and eigen spaces for non-trivial characters vanish here.

So we can pick for each ε generators of these eigen spaces in

$$\eta_{\infty, \varepsilon} \in \left(\bigwedge^{rb_m} \tilde{\mathfrak{p}}_m^* \otimes \mathcal{W}_0(\tau_\infty, \psi_\infty) \otimes M_{\boldsymbol{\eta}}(\mathbf{C}) \right)^{K_m^0},$$

as the latter space is known to give an explicit description of the $(\tilde{\mathfrak{g}}_m, K_m^0)$ -cohomology of τ_∞ . Similarly we have a generator

$$\eta'_\infty \in \left(\bigwedge^{rb_{m'}} \tilde{\mathfrak{p}}_{m'}^* \otimes \mathcal{W}_0(\tau'_\infty, \psi'_\infty) \otimes M_{\boldsymbol{\eta}'}(\mathbf{C}) \right)^{K_{m'}},$$

The relative cohomologies with respect to $K_{m'}$ and $K_{m'}^0$ are here the same due to our assumption on cohomologicality. We apologize for the imprecise notation concerning the additive characters ψ_∞ and ψ'_∞ . We assume them to be compatible with our previous choices, i.e. they are dual to each other in an appropriate sense.

Now we can add local Whittaker functions of finite places to build up global Whittaker functions. To be more precise, choose Maurer-Cartan bases of $\tilde{\mathfrak{p}}_m$ and $\tilde{\mathfrak{p}}_{m'}$ and write with respect to those

$$\eta_{\infty, \varepsilon} = \sum_{\#I=rb_m} \omega_I \otimes w_{\infty, \varepsilon, I},$$

where $w_{\infty, \varepsilon, I} \in \mathcal{W}_0(\tau_\infty, \psi_\infty) \otimes M_{\boldsymbol{\eta}'}(\mathbf{C})$. Then we tensor the latter Whittaker functions with our finite component $w_f \in \mathcal{W}(\tau_f, \psi_f)$ and we get a well defined element

$$\sum_{\#I=rb_m} \omega_I \otimes w_{\varepsilon, I} = \sum_{\#I=rb_m} \omega_I \otimes w_{\infty, \varepsilon, I} \otimes w_f,$$

which by Fourier transform yields a cuspidal class

$$[\sum_{\#I=rb_m} \omega_I \otimes \phi_{\varepsilon, I}] \in H^{rb_m}(\tilde{\mathfrak{g}}_m, K_m^0; L_0^2(G_m(\mathbf{Q}) \backslash G_m(\mathbf{A}_\mathbf{Q}) / K_{m, f}; M_{\boldsymbol{\eta}}(\mathbf{C})))_\varepsilon,$$

for some compact open $K_{m, f} \subseteq G_m(\mathbf{A}_\mathbf{Q}^{(\infty)})$, which in turn yields, thanks to Borel [Bor74, Bor81], a cohomology class with compact support

$$[\eta_\varepsilon] \in H_c^{rb_m}(G_m(\mathbf{Q}) \backslash G_m(\mathbf{A}_\mathbf{Q}) / K_{m, f} K_m^0; \underline{M}_{\boldsymbol{\eta}}(\mathbf{C}))_\varepsilon$$

(note that this transition implicitly requires the formalism we discussed before, and this is eventually compatible with the component action, i.e. the resulting class lies again in the ε -eigen space as desired).

The very same procedure works for η'_∞ and yields a class²

$$[\eta'] \in H_c^{r_{b_{m'}}}(G_{m'}(\mathbf{Q}) \backslash G_{m'}(\mathbf{A}_{\mathbf{Q}}) / K_{m',f} K_{m'}^0; \underline{M}_{\eta'}(\mathbf{C})).$$

Now, depending on whether $m = n$ or not we choose $\alpha_\varepsilon = [\eta_\varepsilon]$ or $\alpha_\varepsilon = [\eta']$, and $\beta_\varepsilon = [\eta']$ or $\beta_\varepsilon = [\eta_\varepsilon]$ respectively. Then we might plug in the universal class

$$\lambda_{\pi,\sigma} := \sum_{\varepsilon} \alpha_\varepsilon \otimes \beta_\varepsilon$$

into our modular symbol and we eventually get the desired periods (for some suitable choice of γ), modulo constants depending on the parity of the critical value, which might vanish. By classical results of Shimura and Manin in the case $n = 2$ and Mazur, Schmidt and Kasten-Schmidt in the case $n = 3$, we know that, depending on the signs of the components of χ_∞ and the parity of ν , there is precisely one ε such that $\alpha_\varepsilon \otimes \beta_\varepsilon$ contributes to the evaluation of the modular symbol at $\lambda_{\pi,\sigma}$ and the corresponding period $\Omega(\sum_{I \cap I' = \emptyset} w_{\infty,\varepsilon,I} \otimes v_{\infty,\varepsilon,I'}, \chi_\infty)$ does not vanish at $\frac{1}{2} + \nu$.

Now rationality and even integrality follows if we choose our local Whittaker vectors in such a way that the resulting class becomes integral, i.e. a non-zero element³ of

$$H_c^{r_{b_m}}(G_m(\mathbf{Q}) \backslash G_m(\mathbf{A}_{\mathbf{Q}}) / K_{m,f} K_m^0; \underline{M}_\eta(\mathcal{O}_E))_\varepsilon, \quad \text{resp.}$$

$$H_c^{r_{b_{m'}}}(G_{m'}(\mathbf{Q}) \backslash G_{m'}(\mathbf{A}_{\mathbf{Q}}) / K_{m',f} K_{m'}^0; \underline{M}_{\eta'}(\mathcal{O}_E)),$$

where \mathcal{O}_E is the ring of integers of a field of rationality E for (π, σ) . It is well known that this choice is possible by virtue of a generalized Eichler-Shimura isomorphism. Eichler-Shimura guarantees that a suitable choice at the finite places yields an integral class modulo scalars. So the essential point in our construction is multiplicity one in bottom degree, i.e. the corresponding eigen spaces are one-dimensional, and an appropriate scaling gives rise to the periods (under the widely believed conjecture that there is a good vector at infinity supporting cohomology; in some sense the work of Kasten-Schmidt can be interpreted as a partial result in this direction in the case $n = 3$; furthermore results of Harder and Mahnkopf also point in this direction, modulo scalars in E).

After all, we have an, up to integral units and possible torsion, well-defined integral element

$$\lambda_{\pi,\sigma} \in H_c^{r_{b_n}}(G_n(\mathbf{Q}) \backslash G_n(\mathbf{A}_{\mathbf{Q}}) / K_{n,f} K_n^0; \underline{M}_\mu(\mathcal{O}_E))$$

$$\otimes_{\mathcal{O}_E} H_c^{r_{b_{n-1}}}(G_{n-1}(\mathbf{Q}) \backslash G_{n-1}(\mathbf{A}_{\mathbf{Q}}) / K_{n-1,f} K_{n-1}^0; \underline{M}_\nu(\mathcal{O}_E)).$$

Extending our modular symbol to the various connected components we get for each component a vector

$$\mathcal{B}_\gamma^{\Gamma_n, \Gamma_{n-1}}(\lambda_{\pi,\sigma}) \in H^0(({}^0G_{n-1})^0; j_\gamma^* M_\mu(\mathcal{O}_E) \otimes_{\mathcal{O}_E} M_\nu(\mathcal{O}_E)),$$

²in this case the component action is trivial.

³eventually the integral structure in cohomology with compact supports induces integral structures on our eigen spaces in Lie algebra cohomology, which in turn as a unique generator up to integral units. We assume that we choose such a generator for any ε .

where

$$j_\gamma : G_{n-1} \rightarrow G_n, \quad g \mapsto \gamma^{-1}j(g),$$

encoding the period integrals from the global Birch Lemma.

4.3 Integral structures on local systems

Consider the $G_n(\mathbf{Q})$ -module $M_\mu(E)$ for any field extension E/\mathbf{Q} . It gives rise to a sheaf $\underline{M}_\mu(E)$ on the locally symmetric space

$$\mathcal{X}_n(K) := G_n(\mathbf{Q}) \backslash G_n(\mathbf{A}_\mathbf{Q}) / K_\infty^0 K,$$

where K is any compact open subgroup of $G_n(\mathbf{A}_\mathbf{Q}^{(\infty)})$ and $K_\infty \subseteq G_{n-1}(\mathbf{R})$ is the product of the standard maximal compact subgroup and the center of $G_{n-1}(\mathbf{R})$. We henceforth assume that for all $g \in G_n(\mathbf{A}_\mathbf{Q}^{(\infty)})$

$$\Gamma_g := G_n(\mathbf{Q}) \cap gKg^{-1}$$

is torsion free. Then $\mathcal{X}_n(K)$ is a manifold and with $\Gamma := \Gamma_1$ we may identify

$$\Gamma \backslash \mathcal{X}_n \subseteq \mathcal{X}_n(K)$$

naturally as the connected component of the the origin. Furthermore this corresponds by strong approximation to the fiber above the identity of the surjective determinant map

$$\det_K : \mathcal{X}_n(K) \rightarrow k^\times \backslash \mathbf{A}_k^\times / (k \otimes_\mathbf{Q} \mathbf{R})^0 \det(K).$$

Note that the base

$$C(K) := k^\times \backslash \mathbf{A}_k^\times / (k \otimes_\mathbf{Q} \mathbf{R})^0 \det(K).$$

is finite. In particular we write

$$\mathcal{X}_n(K)[c] := \det_K^{-1}(c)$$

for the fiber of a class

$$c \in C(K).$$

We often assume that $c \in \mathbf{A}_k^\times$ is also a representative, that we may and do assume to be a finite idèle, i.e. trivial at infinity.

In order to examine integral structures we realize the sheaf $\underline{M}_\mu(E)$ explicitly as

$$\Gamma(U; \underline{M}_\mu(E)) = \{f : G_n(\mathbf{Q})U \rightarrow M_\mu(E) \mid f \text{ locally constant and}$$

$$\forall \gamma_n \in ({}^0G_n)^0(\mathbf{Q}), u \in G_n(\mathbf{Q})U : f(\gamma_n)u = \rho_\mu(\gamma_n)f(u)\}.$$

Now let $\mathcal{O} \subseteq E$ be a subring admitting E as quotient field and let $M_\mu(\mathcal{O})$ be a K -stable \mathcal{O} -lattice in $M_\mu(E)$, i.e.

$$M_\mu(\mathcal{O}) \otimes_\mathcal{O} E \cong M_\mu(E)$$

naturally and

$$\rho_{\boldsymbol{\mu}}(K)(M_{\boldsymbol{\mu}}(\mathcal{O}) \otimes_{\mathbf{Z}} \hat{\mathbf{Z}}) \subseteq M_{\boldsymbol{\mu}}(\mathcal{O}) \otimes_{\mathbf{Z}} \hat{\mathbf{Z}}.$$

We have a sheaf $\underline{M}_{\boldsymbol{\mu}}^0(\mathcal{O})$ on $\mathcal{X}_n(K)[1]$, explicitly given by

$$\begin{aligned} \Gamma(U; \underline{M}_{\boldsymbol{\mu}}^0(\mathcal{O})) &= \{f : \Gamma U \rightarrow M_{\boldsymbol{\mu}}(\mathcal{O}) \mid f \text{ locally constant and} \\ &\quad \forall \gamma \in \Gamma, u \in \Gamma U : f(\gamma)u = \rho_{\boldsymbol{\mu}}(\gamma)f(u)\}. \end{aligned}$$

Any $g \in G_n(\mathbf{A}_{\mathbf{Q}}^{(\infty)})$ induces a diffeomorphism

$$t_g^0 : \mathcal{X}_n(gKg^{-1})[c] \rightarrow \mathcal{X}_n(K)[c \det(g)],$$

by translation

$$G_n(\mathbf{Q})xgKg^{-1} \mapsto G_n(\mathbf{Q})xgK,$$

which only depends on the class gK . Now let

$$gM_{\boldsymbol{\mu}}(\mathcal{O}) := M_{\boldsymbol{\mu}}(E) \cap \rho_{\boldsymbol{\mu}}(g)(M_{\boldsymbol{\mu}}(\mathcal{O}) \otimes_{\mathbf{Z}} \hat{\mathbf{Z}}),$$

where the intersection takes place in

$$M_{\boldsymbol{\mu}}(E) \otimes_{\mathbf{Q}} \mathbf{A}_{\mathbf{Q}}^{(\infty)}.$$

Then this sheaf is stable under gKg^{-1} and for the arithmetic group

$$\Gamma_g := G_n(\mathbf{Q}) \cap gKg^{-1}$$

we have the associated sheaf

$$g\underline{M}_{\boldsymbol{\mu}}^0(\mathcal{O}).$$

on $\mathcal{X}_n(gKg^{-1})[1]$. Then we get the sheaf

$$\underline{M}_{\boldsymbol{\mu}}^0(\mathcal{O})[\det(g)] := t_{g,*}g\underline{M}_{\boldsymbol{\mu}}^0(\mathcal{O})$$

on $\mathcal{X}_n(K)[\det(g)]$. By strong approximation it does only depend on the class of $\det(g)$ in $C(K)$.

Plugging the above sheaves together we get a subsheaf $\underline{M}_{\boldsymbol{\mu}}(\mathcal{O})$ of $\underline{M}_{\boldsymbol{\mu}}(E)$, given by

$$\begin{aligned} \Gamma(U; \underline{M}_{\boldsymbol{\mu}}(\mathcal{O})) &= \{f \in \Gamma(U; \underline{M}_{\boldsymbol{\mu}}(E)) \mid \forall c \in C(K) : \\ &\quad f|_{U \cap \mathcal{X}_n(K)[c]} \in \Gamma(U \cap \mathcal{X}_n(K)[c]; \underline{M}_{\boldsymbol{\mu}}^0(\mathcal{O})[c])\}. \end{aligned}$$

This sheaf may be naturally identified with the topological sum of the sheaves $t_{g_c,*}g_c\underline{M}_{\boldsymbol{\mu}}^0(\mathcal{O})$ for a system of representatives g_c with $\det(g_c) = c$ running through $C(K)$.

In the very same spirit we may for any $g \in G_n(\mathbf{A}_k^{(\infty)})$ identify the pullback of $\underline{M}_{\boldsymbol{\mu}}(\mathcal{O})$ along the translation by g map

$$t_g : \mathcal{X}_n(gKg^{-1}) \rightarrow \mathcal{X}_n(K)$$

with an analogously defined sheaf $\underline{gM}_\mu(\mathcal{O})$, which itself is naturally a subsheaf of $\underline{M}_\mu(E)$ on $\mathcal{X}_n(gKg^{-1})$. To keep track of the various realizations, we write

$$T_g : t_g^* \underline{M}_\mu(\mathcal{O}) \rightarrow \underline{gM}_\mu(\mathcal{O})$$

for the natural isomorphism. Sometimes we consider $t_{g,*}T_g$ also as an isomorphism

$$t_{g,*}T_g : \underline{M}_\mu(\mathcal{O}) \rightarrow t_{g,*}\underline{gM}_\mu(\mathcal{O})$$

of sheaves on $\mathcal{X}_n(K)$. By construction we have a canonical inclusion

$$i_g : \underline{gM}_\mu(\mathcal{O}) \rightarrow \underline{M}_\mu(E).$$

If g turns out to stabilize $M_\mu(\mathcal{O})$ we consider i_g also as an embedding

$$i_g : \underline{gM}_\mu(\mathcal{O}) \rightarrow \underline{M}_\mu(\mathcal{O}),$$

induced by the set-theoretic interpretation of both sides as subsheaves of $\underline{M}_\mu(E)$. For notational efficiency we introduce the abbreviations

$$iT_g := i_g \circ T_g,$$

$$Tt_g^* := T_g \circ t_g^*,$$

and

$$iTt_g^* := i_g \circ T_g \circ t_g^*.$$

We note that

$$(T_x \circ t_x^*) \circ (T_y \circ t_y^*) = T_{xy} \circ t_{xy}^* \tag{9}$$

and similarly for i_g . This formalism carries over to the various kinds of cohomology and integration commutes with T_g and t_g^* and i_g in the appropriate way.

4.4 Hecke operators on cohomology

The action of the Hecke algebra of finite level on cohomology may be most conceptually defined via Hecke correspondences. Another approach comes from the cohomology of groups, by interpreting the direct image with compact supports that occurs in the construction via correspondences as a trace map for groups. We need an explicit description of the action, that we put also in the foreground of our definitions. This also leads directly to the instances of the Eichler-Shimura isomorphism that we need.

Let $K \subseteq G_n(\mathbf{A}_Q^{(\infty)})$ be compact open and pick an element $g \in G_n(\mathbf{A}_Q^{(\infty)})$. Then the double coset represented by g decomposes into finitely many right cosets

$$KgK = \bigsqcup_i g_i K.$$

On the sheaf $\underline{M}_\mu(E)$ on $\mathcal{X}_n(K)$ we define an action sending a section

$$s \in \Gamma(U; \underline{M}_\mu(E))$$

to

$$s|_{[KgK]} := \sum_i iT_{g_i}^*(s)$$

again on the space $\mathcal{X}_n(K)$. This action extends to cohomology, i.e. considering the right derived functors of the global sections functor we get an endomorphism

$$\cdot|_{[KgK]} \in \text{End}_E(H_*^q(\mathcal{X}_n(K); \underline{M}_\mu(E)))$$

for any $*$ in $\{-, c, !\}$.

Now assume that K is of Iwahori level $I_n^{(m)}$ at \mathfrak{p} . Then $\mathcal{H}_{I^{(m)}} \otimes_{\mathbf{Q}} E$ embeds into $\mathcal{H}_E(K, G_n(\mathbf{A}_{\mathbf{Q}}^{(\infty)}))$. Hence as $\mathcal{H}_E(K, G_n(\mathbf{A}_{\mathbf{Q}}^{(\infty)}))$ acts on the cohomology from the left, $\mathcal{H}_{I^{(m)}}$ acts on cohomology from the left and this action is compatible with the action on automorphic forms, i.e. we have a generalized Eichler-Shimura map.

4.5 Cohomological definition of the distribution

Let for any \mathfrak{p} -power ideal \mathfrak{f} in k

$$C(\mathfrak{f}) := k^\times \backslash \mathbf{A}_k^\times / (1 + \mathfrak{f}) \prod_{\mathfrak{q} \nmid \mathfrak{p}} U_{\mathfrak{q}},$$

$$C(\mathfrak{p}^\infty) := \varprojlim C(\mathfrak{f}) = k^\times \backslash \mathbf{A}_k^\times / \prod_{\mathfrak{q} \nmid \mathfrak{p}} U_{\mathfrak{q}},$$

where \mathfrak{f} ranges over the \mathfrak{p} -power ideals. On these groups we have a natural action of the idèles \mathbf{A}_k^\times .

Fix some compact open subgroups K and K' in $\text{GL}_n(\mathbf{A}_k^{(\infty)})$ resp. $\text{GL}_{n-1}(\mathbf{A}_k^{(\infty)})$, which are factorizable and coincide with $I_n^{(m)}$ resp. $I_{n-1}^{(m)}$ at \mathfrak{p} for a fixed $m \geq 1$, and satisfy $j(K') \subseteq K$ and $\det(K) = \det(K')$. Denote Γ resp. Γ' the corresponding arithmetic groups, and assume that for all $g \in G_n(\mathbf{A}_{\mathbf{Q}}^{(\infty)})$ the arithmetic groups

$$G_n(\mathbf{Q}) \cap gKg^{-1}$$

are torsion free. We assume the same statement for K' . We fix $f = \varpi^{\nu_{\mathfrak{p}}(\mathfrak{f})}$, a generator of \mathfrak{f} . Set

$$K(\mathfrak{f}) := j^{-1}(h^{(f)}K(h^{(f)})^{-1}) \cap K'.$$

Then for $\nu_{\mathfrak{p}}(\mathfrak{f}) \geq m$ it is known that the \mathfrak{p} -component of $\det(K(\mathfrak{f}))$ equals $1 + \mathfrak{f}$, cf. [Sch01, Proposition 3.4]. Therefore we have a finite covering map

$$C(K(\mathfrak{f})) \rightarrow C(\mathfrak{f}) \tag{10}$$

with the notation of the previous section.

Let K_∞ be the product of the standard maximal compact subgroup in $G_{n-1}(\mathbf{R})$ and the center of the latter. Consider the locally symmetric space

$$\mathcal{X}(\mathfrak{f}) := \mathrm{GL}_{n-1}(k) \backslash \mathrm{GL}_{n-1}(\mathbf{A}_k) / K(\mathfrak{f}) K_\infty^0.$$

We have a surjection

$$\det_{K(\mathfrak{f})} : \mathcal{X}(\mathfrak{f}) \rightarrow C(K(\mathfrak{f})),$$

induced by the determinant map. The fibers of $\det_{K(\mathfrak{f})}$ are, by strong approximation, given by the translates of

$$\mathcal{X}(\mathfrak{f})[1] = \det_{K(\mathfrak{f})}^{-1}(1) \cong \Gamma(\mathfrak{f}) \backslash \mathcal{X}_{n-1}$$

where $\Gamma(\mathfrak{f}) \subseteq \mathrm{GL}_{n-1}(k)$ is the arithmetic subgroup corresponding to $K(\mathfrak{f})$.

Consider the proper map

$$s_{h(f)} : \mathcal{X}(\mathfrak{f}) \rightarrow \mathcal{X}_n^{\mathrm{ad}}(K),$$

which results from the composition of the embedding j with the translation $t_{h(f)}$ and the adjoint map.

We define topologically for any $h \in G_n(\mathbf{A}_{\mathbf{Q}}^{(\infty)})$ and any $x \in \mathbf{A}_k^{(\infty)\times}$ the topological modular symbol

$$\mathcal{B}_{h,x}^{K,K'} : H_c^{rb_n}(\mathcal{X}_n^{\mathrm{ad}}(K); \underline{M}_\mu(\mathcal{O})) \otimes_{\mathcal{O}} H_c^{rb_{n-1}}(\mathcal{X}_{n-1}^{\mathrm{ad}}(K'); \underline{M}_\nu(\mathcal{O})) \rightarrow$$

$$H^0(\mathcal{X}_{n-1}(j^{-1}(hKh^{-1}) \cap K')[x]; s_h^* \underline{M}_\mu(\mathcal{O}) \otimes_{\mathcal{O}} \underline{M}_\nu(\mathcal{O})),$$

via

$$\lambda = \sum_{\varepsilon} \alpha_\varepsilon \otimes \beta_\varepsilon \mapsto \int_{\mathcal{X}_{n-1}(j^{-1}(hKh^{-1}) \cap K')[x]} \sum_{\varepsilon} s_h^* \alpha_\varepsilon \cup \mathrm{ad}^* \beta_\varepsilon.$$

Assuming that λ is an eigen class for the Hecke operator

$$U_{\mathfrak{p}} := V_{\mathfrak{p}} \otimes V'_{\mathfrak{p}}$$

with eigen value $\kappa_\lambda \in E^\times$ (i.e. λ is of finite slope) we set

$$\kappa_\lambda(\mathfrak{f}) := \kappa_\lambda^{-\nu_{\mathfrak{p}}(\mathfrak{f})},$$

and for any \mathfrak{f} with $\nu_{\mathfrak{p}}(\mathfrak{f}) \geq m$, we considering $x + \mathfrak{f}$ as an element of $C(K(\mathfrak{f}))$, and define

$$\mu_\lambda(x + \mathfrak{f}) := \kappa_\lambda(\mathfrak{f}) \cdot (iT t_{d(x)f t(f)}^* (s_1^*(iT_{h(f)}) \otimes 1) \mathcal{B}_{h(f), x f^{\frac{n(n-1)}{2}}}^{K,K'}(\lambda),$$

as an element of

$$H^0(({}^0G_{n-1})^0; j^* M_\mu(E) \otimes_E M_\nu(E)),$$

the latter space being independent of the levels, and hence independent of \mathfrak{f} and x . To see this observe first that the translation related operators commute with integration. In particular we see that

$$\mu_\lambda(x + \mathfrak{f}) = \kappa_\lambda(\mathfrak{f}) \cdot \int_{\mathcal{X}_{n-1}(d_{(x)}t_{(f)}K(\mathfrak{f})t_{(f)}^{-1}d_{(x)}^{-1})[1]} (iTt_{d_{(x)}ft_{(f)}}^*)(s_1^*(iTt_{h(f)}) \otimes 1) \sum_{\varepsilon} s_1^*(\alpha_\varepsilon) \cup \text{ad}^*(\beta_\varepsilon),$$

and the integrand lives in

$$H_c^{d_{n-1}}(\mathcal{X}_{n-1}(d_{(x)}t_{(f)}K(\mathfrak{f})t_{(f)}^{-1}d_{(x)}^{-1}); s_1^*\underline{M}_\mu(E) \otimes_E \underline{M}_\nu(E)).$$

Now our claim follows as the arithmetic subgroup corresponding to $d_{(x)}t_{(f)}K(\mathfrak{f})t_{(f)}^{-1}d_{(x)}^{-1}$ on the component with determinant 1 is Zariski dense in $({}^0G_{n-1})^0$.

4.6 The distribution relation

In this section we study the effect of the Hecke operator $U_{\mathfrak{p}}$ on our modular symbol and prove the distribution relation if λ is an eigen vector for this operator with non-zero eigen value $\kappa_\lambda \in E^\times$.

To begin with, we need to refine of a known result of Schmidt about the relation of the matrices $h^{(f)}$ and $h^{(f\varpi)}$ (cf. [Sch01, Lemma 3.2] or [Jan11, Lemma B.0.1]).

For any $x \in \mathbf{A}_k^{(\infty)}$ and any $u \in U_n(\mathcal{O}_{\mathfrak{p}})$, $w \in U_{n-1}(\mathcal{O}_{\mathfrak{p}})$ set

$$h(u, w) := t_{(f)}j(w)t_{(f)}^{-1} \cdot h^{(1)} \cdot t_{(f)}u^{-1}t_{(f)}^{-1} \in G_n(\mathcal{O}_{\mathfrak{p}}/\mathfrak{fp}).$$

This defines a group action on a subset of $G_n(\mathcal{O}_{\mathfrak{p}}/\mathfrak{fp})$. Writing $u = (u_{ij})$ and $w = (w_{ij})$ then $h(u, w)$ only depends on the terms

$$u_{12}, u_{23}, \dots, u_{n-1n}$$

and

$$w_{12}, w_{23}, \dots, w_{n-2n-1}.$$

Note that we have for the same reason

$$t_{(f)}u^{-1}t_{(f)}^{-1} \equiv t_{(f)}(-u_{ij})_{ij}t_{(f)}^{-1} \pmod{\mathfrak{fp}}.$$

We conclude that $h(u, w)$ equals the matrix

$$\begin{pmatrix} 0 & \dots & \dots & 0 & fw_{12} & 1 & 1 + fw_{12} - fu_{n-1n} \\ \vdots & & \ddots & fw_{23} & 1 & -fu_{n-2n-1} & 1 + fw_{23} \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & 1 + fw_{34} \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ fw_{n-2n-1} & 1 & -fu_{23} & \ddots & & \vdots & 1 + fw_{n-2n-1} \\ 1 & -fu_{12} & \ddots & & & \vdots & 1 \\ 0 & 0 & \dots & \dots & \dots & 0 & 1 \end{pmatrix}$$

Define the lower triangular unipotent matrices

$$u^- := \begin{pmatrix} 1 & & & \\ fu_{n-2n-1} & 1 & & \\ & \ddots & \ddots & \\ & & fu_{32} & 1 \\ & & & fu_{21} & 1 \end{pmatrix} \in t_{(f)}^{-1} U_{n-1}^-(\mathcal{O}_{\mathfrak{p}}) t_{(f)},$$

and

$$w^- := \begin{pmatrix} 1 & & & \\ -fw_{n-2n-1} & 1 & & \\ & \ddots & \ddots & \\ & & -fw_{32} & 1 \\ & & & -fw_{21} & 1 \end{pmatrix} \in t_{(f)}^{-1} U_{n-1}^-(\mathcal{O}_{\mathfrak{p}}) t_{(f)}.$$

Then the first $n-1$ columns of the matrix

$$u^- \cdot h(u, w) \cdot w^-$$

equal those of $h^{(1)}$ and the last column is the tranpose of

$$(1 + fw_{12} - fu_{n-1n}, 1 + fw_{23} - fu_{n-2n-1}, \dots, 1 + fw_{n-2n-1} - fu_{23}, 1 - fu_{12}, 1).$$

Set

$$d(u, w) := \text{diag}(1 - fu_{12}, 1 + fw_{n-2n-1} - fu_{23}, \dots, 1 + fw_{12} - fu_{n-1n})$$

and

$$d'(u, w) := \text{diag}(1 - fw_{12} + fu_{n-1n}, \dots, 1 - fw_{n-2n-1} + fu_{23}, 1 + fu_{12}).$$

Note that

$$\det(d(u, w)) \cdot \det(d'(u, w)) = 1.$$

Lemma 4.1. *For any u, w as above there exist matrices $k'_{u,w} \in I_{n-1}^{(m)}$ and $k_{u,w} \in I_n^{(m)}$ with the property that*

$$t_{(\varpi)}^{-1} j(w) \cdot h^{(f)} \cdot u^{-1} t_{(\varpi)} = j(k') \cdot h^{(f\varpi)} \cdot k^{-1}$$

and that furthermore sending u, w to

$$\det(k_{u,w}) = \det(k'_{u,w}) \pmod{\mathfrak{fp}}$$

defines an epimorphism of groups

$$U_n(\mathcal{O}_{\mathfrak{p}})/t_{(\varpi)} U_n(\mathcal{O}_{\mathfrak{p}}) t_{(\varpi)}^{-1} \times U_{n-1}(\mathcal{O}_{\mathfrak{p}})/t_{(\varpi)} U_{n-1}(\mathcal{O}_{\mathfrak{p}}) t_{(\varpi)}^{-1} \rightarrow 1 + \mathfrak{f}/1 + \mathfrak{fp}.$$

Proof. By the above discussion there is a matrix $n \in I_n^{(m)}$ with

$$n \equiv \mathbf{1}_n \pmod{\mathfrak{fp}}$$

and

$$j(d'(u, w)) \cdot u^- \cdot h(u, w) \cdot w^- \cdot d(u, w) \cdot n = h^{(1)}.$$

Now observe that

$$t_{(\varpi)}^{-1} \cdot j(w) \cdot h^{(f)} \cdot u^{-1} \cdot t_{(\varpi)} = t_{(f\varpi)}^{-1} \cdot h(u, w) \cdot t_{(f\varpi)}.$$

Therefore the choice

$$k'_{u,w} := t_{(f\varpi)}^{-1} \cdot j(d'(u, w)) \cdot u^- \cdot t_{(f\varpi)} \in I_{n-1}^{(m)}$$

$$k_{u,w} := \left(t_{(f\varpi)}^{-1} \cdot j(w^- \cdot d(u, w)) \cdot n \cdot t_{(f\varpi)} \right)^{-1} \in I_n^{(m)}$$

proves the claim. \square

As an application we have

Lemma 4.2. *For any*

$$\alpha \in H_c^{rb_n}(\mathcal{X}_n^{\text{ad}}(K); \underline{M}_\mu(E))$$

and any

$$\beta \in H_c^{rb_{n-1}}(\mathcal{X}_{n-1}^{\text{ad}}(K'); \underline{M}_\nu(E))$$

and any $u \in U_n(\mathcal{O}_{\mathfrak{p}})$ and $w \in U_{n-1}(\mathcal{O}_{\mathfrak{p}})$ we have

$$\begin{aligned} & (s_1^*(iT_{h(f)}) \otimes 1) \mathcal{B}_{h(f), x f^{\frac{n(n-1)}{2}}}^{ut(\varpi) K(ut(\varpi))^{-1}, wt(\varpi) K'(wt(\varpi))^{-1}} (iT_{ut(\varpi)}^* \alpha \otimes iT_{w\varpi t(\varpi)}^* \beta) = \\ & iT_{d(\det(k_{u,w-1})) \varpi t(\varpi)}^* \circ (s_1^* iT_{h(f\varpi)} \otimes 1) \mathcal{B}_{h(f\varpi), x \det(k_{u,w-1})(f\varpi)^{\frac{n(n-1)}{2}}}^{K, K'} (\alpha \otimes \beta) \end{aligned}$$

Proof. Under the above hypothesis we have

$$\begin{aligned} & \mathcal{B}_{h(f), x f^{\frac{n(n-1)}{2}}}^{ut(\varpi) K(ut(\varpi))^{-1}, wt(\varpi) K'(wt(\varpi))^{-1}} (iT_{ut(\varpi)}^* \alpha \otimes iT_{w\varpi t(\varpi)}^* \beta) = \\ & \int s_{h(f)}^* (iT_{ut(\varpi)}^* \alpha) \cup \text{ad}^* (iT_{w\varpi t(\varpi)}^* \beta). \\ & \mathcal{X}_{n-1}(j^{-1}(h(f) ut(\varpi) K(h(f) ut(\varpi))^{-1}) \cap wt(\varpi) K'(wt(\varpi))^{-1}) [x f^{\frac{n(n-1)}{2}}] \end{aligned}$$

Now due to (9) we have an identity of pullbacks

$$iT_{(w\varpi t(\varpi))^{-1}}^* = iT_{(\varpi t(\varpi))^{-1}}^*,$$

because $w \in {}^0G_{n-1}$. Hence the composition of this pullback with $s_1^* iT_{h(f)} \otimes 1$ maps the above section to

$$(s_1^* iT_{t_{(\varpi)}^{-1}j(w)^{-1}h(f)ut_{(\varpi)}} \otimes 1) \int_{\mathcal{X}_{n-1}(j^{-1}(t_{(\varpi)}^{-1}j(w)^{-1}h(f)ut_{(\varpi)})K(t_{(\varpi)}^{-1}j(w)^{-1}h(f)ut_{(\varpi)})^{-1}) \cap K'} [x(f\varpi)^{\frac{n(n-1)}{2}}]} s_{t_{(\varpi)}^{-1}j(w)^{-1}h(f)ut_{(\varpi)}}^* \alpha \cup \text{ad}^* \beta.$$

Now by Lemma 4.1 we have the elements $k' = k'_{u^{-1},w} \in K'$ and $k = k_{u^{-1},w} \in K$ with $\det(k') = \det(k) \in 1 + \mathfrak{f}$, such that

$$j(k')^{-1} \cdot h^{(f\varpi)} \cdot k = t_{(\varpi)}^{-1}j(w)^{-1} \cdot h^{(f)} \cdot ut_{(\varpi)}.$$

Consequently the pullback $iTt_{k'}^*$ maps the above section to

$$(s_1^* iT_{h(f\varpi)} \otimes 1) \int_{\mathcal{X}_{n-1}(j^{-1}(h^{(f\varpi)}K(h^{(f\varpi)})^{-1}) \cap K') [x \det(k)^{-1}(f\varpi)^{\frac{n(n-1)}{2}}]} s_{h^{(f\varpi)}}^* (iTt_k^* \alpha) \cup \text{ad}^* (iTt_{k'}^* \beta).$$

We have

$$iTt_k^*(\alpha) = \alpha$$

and similarly for β . Now we observe that

$$d_{(\det(k)^{-1})} \cdot k' \in ({}^0G_{n-1})^0,$$

as this element has determinant equal to 1. Hence

$$iTt_{k'}^* = iTt_{d_{(\det(k))}}^*,$$

and the claim follows. \square

We are now in a position to prove

Theorem 4.3. *For any $x \in \mathbf{A}_k^{(\infty)}$ and any ideal $\mathfrak{f} = \mathfrak{p}^{\nu_{\mathfrak{p}}(\mathfrak{f})}$ with $\nu_{\mathfrak{p}}(\mathfrak{f}) \geq m$ we have*

$$\mu_{\lambda}(x + \mathfrak{f}) = \sum_{a \pmod{\mathfrak{p}}} \mu_{\lambda}(x + af + \mathfrak{f}\mathfrak{p}).$$

In particular μ_{λ} is a distribution on $C(K(\mathfrak{p}^{\infty})) = \varprojlim_{\mathfrak{f}} C(K(\mathfrak{f}))$ with values in

$$H^0(({}^0G_{n-1})^0; j^* M_{\mu}(E) \otimes_E M_{\nu}(E)).$$

Proof. By our hypothesis λ is an eigen vector for

$$U_{\mathfrak{p}} = V_{\mathfrak{p}} \otimes V'_{\mathfrak{p}},$$

which by Lemma 1.2 means that

$$\kappa_{\lambda} \cdot \lambda = \sum_{\varepsilon, u, w} (iTt_{ut_{(\varpi)}}^* \alpha_{\varepsilon}) \otimes (iTt_{w\varpi t_{(\varpi)}}^* \beta_{\varepsilon}),$$

where both sides are interpreted as cohomology classes with respect to the intersections \tilde{K} resp. \tilde{K}' of the compact open subgroups $ut_{(\varpi)}K(ut_{(\varpi)})^{-1}$ for all u and $wt_{(\varpi)}K'(wt_{(\varpi)})^{-1}$ for all w respectively. By Lemmata 4.1 and 4.2 this yields the decomposition

$$\begin{aligned} & [K(\mathfrak{f}) : K(\mathfrak{fp})] \cdot (s_1^* iT_{h(f)} \otimes 1) \int_{\mathcal{X}(\mathfrak{f})[xf^{\frac{n(n-1)}{2}}]} (s_{h(f)}^* \alpha \cup \text{ad}^* \beta) = \\ & \frac{(U_n(\mathcal{O}_{k,\mathfrak{p}}) : t_{(\varpi)}U_n(\mathcal{O}_{k,\mathfrak{p}})t_{(\varpi)}^{-1}) \cdot (U_{n-1}(\mathcal{O}_{k,\mathfrak{p}}) : t_{(\varpi)}U_{n-1}(\mathcal{O}_{k,\mathfrak{p}})t_{(\varpi)}^{-1})}{\mathfrak{N}(\mathfrak{p})} \cdot \\ & \kappa_\lambda \cdot \sum_{a \pmod{\mathfrak{p}}} iT_{\varpi t_{(\varpi)}}^* \circ (s_1^* iT_{h(f\varpi)} \otimes 1) \int_{\mathcal{X}(\mathfrak{fp})[(x+af)(f\varpi)^{\frac{n(n-1)}{2}}]} (s_{h(f\varpi)}^* \alpha \cup \text{ad}^* \beta). \end{aligned}$$

By the following known index formulas

$$(I_{n-1}^{(m)} : K(\mathfrak{f})) = \mathfrak{N}(\mathfrak{f})^{\frac{(n+1)n(n-1)+n(n-1)(n-2)}{6}} \quad (11)$$

cf. [Sch01, Lemmata 3.6 and 3.7], and

$$(U_n(\mathcal{O}_{k,\mathfrak{p}}) : t_{(f)}U_n(\mathcal{O}_{k,\mathfrak{p}})t_{(f)}^{-1}) = \mathfrak{N}(\mathfrak{f})^{\frac{(n+1)n(n-1)}{6}},$$

cf. [KMS00, Proof of Lemma 3.2], the claim follows. \square

4.7 Boundedness in the ordinary case

Let E be a number field. Fix an embedding $i_{\mathfrak{p}} : E \rightarrow \bar{k}_{\mathfrak{p}}$ and denote by $E_{\mathfrak{p}}$ the closure of the image. Similarly $\mathcal{O}_{E,\mathfrak{p}}$ denotes the closure of $i_{\mathfrak{p}}(\mathcal{O}_E)$. We write $\|\cdot\|_{\mathfrak{p}}$ for the norm on E induced by $i_{\mathfrak{p}}$. Then if λ is an eigen vector for $\kappa_\lambda \in E$ we say that λ is *ordinary at \mathfrak{p}* if

$$\|\kappa_\lambda\|_{\mathfrak{p}} = \mathfrak{N}(\mathfrak{p})^{-\nu_{\min} \cdot \frac{n(n-1)}{2}}. \quad (12)$$

We say that λ is *of finite slope* if $\kappa_\lambda \neq 0$ and then the integer

$$\nu_{\mathfrak{p}} \left(\kappa_\lambda \cdot \varpi^{-\nu_{\min} \cdot \frac{n(n-1)}{2}} \right) \in \mathbf{Z}$$

is called the *slope* of λ at \mathfrak{p} . So λ is ordinary if and only if it is of slope 0.

We know that $M_{\mu}(\mathcal{O}_{E,\mathfrak{p}})$ and $M_{\nu}(\mathcal{O}_{E,\mathfrak{p}})$ are stable under $G_n(\mathbf{A}_{\mathbf{Q}}^{(\infty p)})$ resp. $G_{n-1}(\mathbf{A}_{\mathbf{Q}}^{(\infty p)})$. We assume that

$$j^* M_{\mu}(\mathcal{O}_{E,\mathfrak{p}}) \otimes_{\mathcal{O}_{E,\mathfrak{p}}} M_{\nu}(\mathcal{O}_{E,\mathfrak{p}}) \otimes_{\mathcal{O}_{E,\mathfrak{p}}} M_{(-\nu_{\min})}(\mathcal{O}_{E,\mathfrak{p}}) \quad (13)$$

is stable under \mathfrak{p} -integral matrices, i.e. under the action of

$$G_{n-1,\mathfrak{p}} := \{g \in \text{GL}_{n-1}(k_{\mathfrak{p}}) \mid g \in \mathcal{O}_{k,\mathfrak{p}}^{n-1 \times n-1}\}.$$

Integral models with this property clearly exist, as we know from section 4.1 that the module (13) decomposes into

$$\bigoplus_{\nu \in \text{Emb}(\check{\nu}, \mu)} M_{\check{\nu}} \otimes M_{\nu}[\nu - \nu_{\min}]$$

and $\nu \geq \nu_{\min}$.

Theorem 4.4. *If λ is ordinary at \mathfrak{p} , then μ_{λ} takes values in*

$$H^0(({}^0G_{n-1})^0; j^* M_{\mu}(\mathcal{O}_{E, \mathfrak{p}}) \otimes_{\mathcal{O}_{E, \mathfrak{p}}} M_{\nu}(\mathcal{O}_{E, \mathfrak{p}})).$$

Proof. First we observe that

$$d_{(x)} \cdot t_{(f)} \cdot h^{(f)} = d_{(x)} \cdot h^{(1)} \cdot t_{(f)}.$$

We might choose $x \in \mathbf{A}_k^{(\infty) \times}$ in such a way that $x_v \in \mathcal{O}_{k, v}^{\times}$ for all $v \mid p$. Now $d_{(x)} f t_{(f)}$ acts on $M_{(-\nu_{\min})}(\mathcal{O}_{E, \mathfrak{p}})$ via multiplication by

$$f_0 := \left(x f^{\frac{n(n-1)}{2}} \right)^{-\nu_{\min}}.$$

By our choice of x and the ordinarity condition (12)

$$\|\kappa_{\lambda}(\mathfrak{f}) f_0^{-1}\|_{\mathfrak{p}} = 1.$$

We deduce that we have an identity

$$\kappa_{\lambda}(\mathfrak{f}) \cdot \underline{M}_{(-\nu_{\min})}^{-1}(d_{(x)} f t_{(f)})^{-1}(\mathcal{O}_{E, \mathfrak{p}}) = \underline{M}_{(-\nu_{\min})}(\mathcal{O}_{E, \mathfrak{p}})$$

of subsheaves of $\underline{M}_{(-\nu_{\min})}(E_{\mathfrak{p}})$. In particular we have the integral global section

$$\gamma := \kappa_{\lambda}(\mathfrak{f}) \cdot T t_{(d_{(x)} f t_{(f)})^{-1}}^* 1 \in M_{(-\nu_{\min})}(\mathcal{O}_{E, \mathfrak{p}}). \quad (14)$$

Consider the natural isomorphism

$$r : H^0(\mathcal{X}_{n-1}(d_{(x)} t_{(f)} K(\mathfrak{f}) t_{(f)}^{-1} d_{(x)}^{-1})[1]; s_1^* \underline{M}_{\mu}(E) \otimes_E \underline{M}_{\nu}(E))[-\nu_{\min}] \rightarrow$$

$$H^0(\mathcal{X}_{n-1}(d_{(x)} t_{(f)} K(\mathfrak{f}) t_{(f)}^{-1} d_{(x)}^{-1})[1]; s_1^* \underline{M}_{\mu}(E) \otimes_E \underline{M}_{\nu}(E))[-\nu_{\min}],$$

which respects integral structures in the obvious way and commutes with integration. In particular for any choice of cohomology classes

$$\alpha \otimes \beta \in H_c^{r b_n}(\mathcal{X}_n^{\text{ad}}(K); \underline{M}_{\mu}(\mathcal{O}_E)) \otimes_{\mathcal{O}_E} H_c^{r b_n - 1}(\mathcal{X}_{n-1}^{\text{ad}}(K'); \underline{M}_{\nu}(\mathcal{O}_E))$$

the element

$$\kappa_{\lambda}(\mathfrak{f}) \cdot \left(T t_{d_{(x)} f t_{(f)}}^* (s_1^* T_{h(f)} \otimes 1) \int_{\mathcal{X}(\mathfrak{f})[x f^{\frac{n(n-1)}{2}}]} s_{h(f)}^* \alpha \cup \text{ad}^* \beta \right) \otimes 1$$

maps, by (14), under r to

$$Tt_{d(x)ft(f)}^*(s_1^*T_{h(f)} \otimes 1 \otimes 1) \int (s_{h(f)}^* \alpha \cup (\text{ad}^* \beta \otimes \gamma)) \in H^0(\mathcal{X}_{n-1}(d_{(x)}t_{(f)}K(\mathfrak{f})t_{(f)}^{-1}d_{(x)}^{-1})[1]; \mathcal{X}(\mathfrak{f})[xf^{\frac{n(n-1)}{2}}]) \quad (15)$$

$$s_1^* \underline{d_{(x)}h^{(1)}t_{(f)}M}_{\mu}(\mathcal{O}_{E,\mathfrak{p}}) \otimes \underline{d_{(x)}ft_{(f)}M}_{\nu}(\mathcal{O}_{E,\mathfrak{p}}) \otimes \underline{d_{(x)}ft_{(f)}M}_{(-\nu_{\min})}(\mathcal{O}_{E,\mathfrak{p}}).$$

Now as

$$d_{(x)}f \cdot t_{(f)} \in G_{n-1,\mathfrak{p}},$$

the module (13) is stable under this element, in particular the section (15) lies in

$$H^0(\mathcal{X}(d_{(x)}t_{(f)}K(\mathfrak{f})t_{(f)}^{-1}d_{(x)}^{-1})[1]; s_1^* \underline{d_{(x)}h^{(1)}d_{(x)}^{-1}M}_{\mu}(\mathcal{O}_{E,\mathfrak{p}}) \otimes \underline{M}_{\nu}(\mathcal{O}_{E,\mathfrak{p}}) \otimes \underline{M}_{(-\nu_{\min})}(\mathcal{O}_{E,\mathfrak{p}})).$$

As the isomorphism r respects integral structures, we conclude that

$$\mu(x+\mathfrak{f}) \in H^0(\mathcal{X}_{n-1}(d_{(x)}t_{(f)}K(\mathfrak{f})t_{(f)}^{-1}d_{(x)}^{-1})[1]; s_1^* \underline{d_{(x)}h^{(f)}d_{(x)}^{-1}M}_{\mu}(\mathcal{O}_{E,\mathfrak{p}}) \otimes \underline{M}_{\nu}(\mathcal{O}_{E,\mathfrak{p}})).$$

Finally, due to our choice of x we have

$$d_{(x)} \cdot h^{(1)} \cdot d_{(x)}^{-1} \in K \cdot G_n(\mathbf{A}_{\mathbf{Q}}^{(\infty p)}),$$

hence $M_{\mu}(\mathcal{O}_{E,\mathfrak{p}})$ is stable under this matrix, concluding the proof. \square

We remark that the same proof yields an explicit bound on the order of the resulting distribution in the case of positive finite slope.

4.8 The interpolation formula

Now let π and σ denote irreducible cuspidal automorphic representations of GL_n and GL_{n-1} over k respectively, possessing non-zero $I_n^{(m)}$ resp. $I_{n-1}^{(m)}$ -invariant vectors at \mathfrak{p} . Assume that π and σ are regular algebraic with cohomological coefficients as in section 4.1. Then their finite parts $\pi^{(\infty)}$ and $\sigma^{(\infty)}$ are defined over a number field $E = \mathbf{Q}(\pi, \sigma)$ [Clo90, Théorème 3.13 resp. Proposition 3.16].

Note that the Hecke polynomial $H_{\mathfrak{p}}$ eventually lies in $\mathcal{H}_{I_n^{(m)}}[X]$. Choose Hecke roots

$$\lambda_1, \dots, \lambda_n \in E,$$

for $\pi_{\mathfrak{p}}$ in the sense of (2), i.e. $H_{\mathfrak{p}}(\lambda_{\nu})$ annihilate a non-zero vector $w_{\mathfrak{p}}^0$ in the Whittaker model of $\pi_{\mathfrak{p}}$, and similarly Hecke roots

$$\lambda'_1, \dots, \lambda'_{n-1} \in E,$$

for $\sigma_{\mathfrak{p}}$ annihilating a vector $v_{\mathfrak{p}}^0$. Let

$$\underline{\lambda} := (\lambda_1, \dots, \lambda_{n-1}) \in E^{n-1},$$

and

$$\underline{\lambda}' := (\lambda_1, \dots, \lambda_{n-1}) \in E^{n-1}.$$

We remark that we include one more eigen value in $\underline{\lambda}'$ as in the case of trivial central character. We set

$$\lambda'' := (\lambda'_1, \dots, \lambda'_{n-2}).$$

With this notation let

$$\kappa_{\underline{\lambda}} := \mathfrak{N}(\mathfrak{p})^{-\frac{n(n-1)(n-2)}{6}} \cdot \prod_{\nu=1}^{n-1} \lambda_{\mathfrak{p},\nu}^{n-\nu}.$$

and

$$\kappa_{\underline{\lambda}'} := \mathfrak{N}(\mathfrak{p})^{-\frac{n(n-1)(n-2)}{6}} \cdot \prod_{\nu=1}^{n-1} \lambda_{\mathfrak{p},\nu}'^{n-\nu}.$$

We call the tuple $(\pi, \sigma, \underline{\lambda}, \underline{\lambda}')$ of *finite slope at \mathfrak{p}* if the following three conditions hold:

- (i) $V_{\mathfrak{p},n-1}$ acts on $\sigma_{\mathfrak{p}}$ via the scalar

$$\eta_{n-1} = \mathfrak{N}(\mathfrak{p})^{-\frac{(n-1)(n-2)}{2}} \cdot \prod_{\nu=1}^{n-1} \lambda'_{\nu}. \quad (16)$$

- (ii) The vectors $w_{\mathfrak{p}}^0$ and $v_{\mathfrak{p}}^0$ may be chosen in such a way that

$$\Pi_{\underline{\lambda}}^0(w_{\mathfrak{p}}^0)(\mathbf{1}_n) = \Pi_{\underline{\lambda}''}^0(v_{\mathfrak{p}}^0)(\mathbf{1}_{n-1}) = \prod_{\nu=1}^{n-1} (1 - \mathfrak{N}(\mathfrak{p})^{-\nu}).$$

- (iii) The *slope*

$$\nu_{\mathfrak{p}} \left(\kappa_{\underline{\lambda}} \cdot \kappa_{\underline{\lambda}'} \cdot \varpi^{-\nu_{\min}, \frac{n(n-1)}{2}} \right) \in \mathbf{Z} \cup \{\infty\},$$

is finite (i.e. λ_n might well be zero).

If in addition the slope is 0, we call the datum *ordinary*.

Assuming that the Whittaker vectors satisfy condition (ii), we set

$$\tilde{w}_{\mathfrak{p}} := \Pi_{\underline{\lambda}}^0(w_{\mathfrak{p}}^0),$$

and

$$\tilde{v}_{\mathfrak{p}} := \Pi_{\underline{\lambda}}^0(v_{\mathfrak{p}}^0).$$

Assume that the cohomology class $\lambda_{\pi,\sigma}$ was constructed as in section 4.2, where the local Whittaker vectors at \mathfrak{p} were chosen as $w_{\mathfrak{p}}^0$ and $v_{\mathfrak{p}}^0$ as above. We define

$$\tilde{\lambda}_{\pi,\sigma,\underline{\lambda},\underline{\lambda}'} := \Pi_{\underline{\lambda}}^0 \otimes \Pi_{\underline{\lambda}''}^0(\lambda_{\pi,\sigma}).$$

By Proposition 1.3 and the condition on the action of $V_{\mathfrak{p},n-1}$ on $\sigma_{\mathfrak{p}}$, this is an eigen vector for the Hecke operator $U_{\mathfrak{p}}$ with eigen value $\kappa_{\underline{\lambda}} \cdot \kappa_{\underline{\lambda}'}$. Then if $\underline{\lambda}$ and $\underline{\lambda}'$ are ordinary, so is $\tilde{\lambda}_{\pi,\sigma,\underline{\lambda},\underline{\lambda}'}$.

We remark that this modification is compatible with modification on the automorphic side, i.e. $\tilde{\lambda}_{\pi,\sigma,\underline{\lambda},\underline{\lambda}'}$ corresponds to the collection of automorphic forms $\tilde{\phi}_{\iota}$ and $\tilde{\varphi}_{\iota}$, which are constructed by applying $\Pi_{\underline{\lambda}}^0$ resp. $\Pi_{\underline{\lambda}'}^0$ to the automorphic forms ϕ_{ι} resp. φ_{ι} corresponding to $\lambda_{\pi,\sigma}$.

The following theorem strengthens and generalizes [Jan11, Theorem 4.4].

Theorem 4.5. *Assume that $(\pi, \sigma, \underline{\lambda}, \underline{\lambda}')$ is of finite slope. Then for any character $\chi : k^{\times} \backslash \mathbf{A}_k^{\times} \rightarrow \mathbf{C}^{\times}$ of finite order with non-trivial \mathfrak{p} -power conductor \mathfrak{f}_{χ} , we have the interpolation formula*

$$\tau \circ \int_{C(\mathfrak{p}^{\infty})} \chi d\mu_{\tilde{\lambda}_{\pi,\sigma,\underline{\lambda},\underline{\lambda}'}} = \left(\Omega(w_{\infty} \otimes v_{\infty}, \chi_{\infty}) \left(\frac{1}{2} + \nu \right) \cdot \hat{\kappa}^{\nu}(\mathfrak{f}_{\chi}) \cdot G(\chi)^{\frac{n(n-1)}{2}} \cdot L\left(\frac{1}{2} + \nu, (\pi \times \sigma) \otimes \chi\right) \right)_{\nu \in \text{Emb}(\tilde{\nu}, \mu)}.$$

Here $\hat{\kappa}^{\nu}(\mathfrak{f}_{\chi})$ is given explicitly by

$$\hat{\kappa}^{\nu}(\mathfrak{f}_{\chi}) := \mathfrak{N}(\mathfrak{f}_{\chi})^{\frac{n(n-1)(n-2)}{6} + (\nu - \nu_{\min}) \frac{n(n-1)}{2}} \cdot (\kappa_{\underline{\lambda}} \cdot \kappa_{\underline{\lambda}'})^{-\nu_{\mathfrak{p}}}(\mathfrak{f}_{\chi}).$$

For $\pi_{\mathfrak{p}}$ resp. $\sigma_{\mathfrak{p}}$ spherical, with pairwise distinct Hecke roots, it is well known that the corresponding data are all of finite slope (cf. [KMS00, Proposition 4.12]).

Up to the computation of the Euler factors at the finite places $v \nmid \mathfrak{p}$ where $\det(K) \neq \mathcal{O}_v^{\times}$, Theorem 4.5 immediately generalizes to arbitrary finite order characters of $C(K(\mathfrak{p}^{\infty}))$ with \mathfrak{p} in its conductor, cf. (10).

Proof. We may choose \mathfrak{f} in such a way that $\nu_{\mathfrak{p}}(\mathfrak{f}) \geq m$ and $\mathfrak{f}_{\chi} \mid \mathfrak{f}$. Then

$$\int_{C(\mathfrak{p}^{\infty})} \chi d\mu_{\tilde{\lambda}_{\pi,\sigma}} = \sum_{x \in C(\mathfrak{f})} \chi(x) \mu_{\tilde{\lambda}_{\pi,\sigma,\underline{\lambda},\underline{\lambda}'}}(x + \mathfrak{f}),$$

and the ν -th component, after composing with τ , is (up to computable indices)

$$\sum_{\iota, x \in C(\mathfrak{f})} \chi(x) \int_{\tilde{C}_{\mathfrak{f}}} \tilde{\phi}_{\iota} \left(j(gd_{(x)} f t_f) \cdot h^{(f)} \right) \cdot \tilde{\varphi}_{\iota}(gd_{(x)} f t_f) \|\det(gd_{(x)} f t_f)\|^{\nu} dg,$$

by the description of the period integrals given in section 4.1. Writing $C_{\mathfrak{f}}[f_{\chi}^{\frac{n(n-1)}{2}}]$ for the corresponding fiber with determinant $f_{\chi}^{\frac{n(n-1)}{2}}$ the right invariance of the Haar measure yields that the period integral in question equals

$$\mathfrak{N}(\mathfrak{f}_{\chi}^{-1})^{-\nu \frac{n(n-1)}{2}} \int_{C_{\mathfrak{f}}[f_{\chi}^{\frac{n(n-1)}{2}}]} \tilde{\phi}_{\iota} \left(j(gd_{(x)}) t_{(f f_{\chi}^{-1})} h^{(f)} \right) \tilde{\varphi}_{\iota}(gd_{(x)} f f_{\chi}^{-1} t_{(f f_{\chi}^{-1})}) \|\det(gd_{(x)})\|^{\nu} dg.$$

Now $\tilde{v}_{\mathfrak{p}}$ is an eigen vector for the operator $V'_{\mathfrak{p}}$ with eigen value $\kappa_{\underline{\lambda}'}$, furthermore $\psi_{\mathfrak{p}}$ is unramified, and therefore, writing $\delta := \nu_{\mathfrak{p}}(\mathfrak{ff}_{\chi}^{-1})$,

$$\begin{aligned} v_{\mathfrak{p}}(ff_{\chi}^{-1}t_{(ff_{\chi}^{-1})}) &= \mathfrak{N}(\mathfrak{ff}_{\chi}^{-1})^{-\frac{n(n-1)}{2}} \cdot V'_{\mathfrak{p}}{}^{\delta} v(\mathbf{1}_{n-1}) = \\ \mathfrak{N}(\mathfrak{ff}_{\chi}^{-1})^{-\frac{(n-1)(n-2)}{2}} \cdot \sum_u v_{\mathfrak{p}}(uf f_{\chi}^{-1}t_{(ff_{\chi}^{-1})}) &= \mathfrak{N}(\mathfrak{ff}_{\chi}^{-1})^{\frac{(n-1)(n-2)}{2}} \cdot \kappa_{\underline{\lambda}'}^{\delta} \cdot v_{\mathfrak{p}}(\mathbf{1}_{n-1}), \end{aligned}$$

where $u \in U_{n-1}(\mathcal{O}_{\mathfrak{p}})$ runs through a system of representatives in the sense of Lemma 1.2. An analogous argument applies to $\tilde{w}_{\mathfrak{p}}$ and Corollary 2.5, together with the known index (11), then concludes the proof. \square

5 The functional equation

In order to establish the functional equation, we need to introduce compatible notions of contragredience in various settings. This formalism is more involved than in the classically known low-dimensional case $n = 2$.

5.1 Contragredient Hecke modules

We use the notation of section 1 and start with considering the full level. Consider the twisted main involution

$$\iota : g \mapsto w_n g^{-t} w_n$$

of GL_n , where the superscript $-t$ denotes matrix inversion composed with transpose. This is an outer automorphism of order 2. Let \mathcal{M} be a vector space over a field E with a left action of the Hecke algebra \mathcal{H}_I of Iwahori level $I_n^{(m)}$.

We consider the full Hecke algebra \mathcal{H}_G as embedded into \mathcal{H}_I . Now as ι stabilizes the corresponding Hecke pairs, it induces outer automorphisms of \mathcal{H}_G and \mathcal{H}_I , commuting with the embedding. It also stabilizes the pair (K_B, B) and commutes with the embeddings into \mathcal{H}_B .

We have the ι -twisted \mathcal{H}_I -module \mathcal{M}^{\vee} . It comes with a canonical map

$$\mathcal{M} \rightarrow \mathcal{M}^{\vee}, \quad m \mapsto m^{\vee},$$

which is twisted \mathcal{H}_I -invariant.

Let $m \in \mathcal{M}$. T_n acts on m by a scalar $c \in E^{\times}$, then T_n acts on

$$m^{\vee} \in \mathcal{M}^{\vee}$$

via the inverse scalar c^{-1} . More generally, the action of the Hecke operator T_{ν} on \mathcal{M}^{\vee} is given by

$$T_{\nu} m^{\vee} = T_n(T_{n-\nu} m)^{\vee}.$$

Using this relation we get

Proposition 5.1. *If for some $\lambda \in E^\times$*

$$H_{\mathfrak{p}}(\lambda)m = 0,$$

then

$$\lambda^\vee := \mathfrak{N}(\mathfrak{p})^{n-1}\lambda^{-1}$$

is a Hecke root for m^\vee , i.e.

$$H_{\mathfrak{p}}(\lambda^\vee)m^\vee = 0.$$

Proof. If we have for some $\lambda \in E^\times$

$$H_{\mathfrak{p}}(\lambda)m = 0,$$

then

$$\begin{aligned} H_{\mathfrak{p}}(\mathfrak{N}(\mathfrak{p})^{n-1}\lambda^{-1})m^\vee &= \sum_{\nu=0}^n (-1)^\nu \mathfrak{N}(\mathfrak{p})^{\frac{(\nu-1)\nu}{2}} (\mathfrak{N}(\mathfrak{p})^{n-1}\lambda^{-1})^{n-\nu} T_\nu m^\vee = \\ &= \mathfrak{N}(\mathfrak{p})^{\frac{n(n-1)}{2}} \lambda^{-n} \sum_{\nu=0}^n \sum_{\nu=0}^n (-1)^\nu \mathfrak{N}(\mathfrak{p})^{\frac{(\nu-1)\nu}{2} + \frac{n(n-1)}{2} - \nu(n-1)} \lambda^\nu T_n(T_{n-\nu}m)^\vee. \end{aligned}$$

Now an easy calculation shows that

$$\frac{(\nu-1)\nu}{2} + \frac{n(n-1)}{2} - \nu(n-1) = \frac{(n-\nu-1)(n-\nu)}{2}. \quad (17)$$

Therefore

$$\mathfrak{N}(\mathfrak{p})^{\frac{n(n-1)}{2}} (-\lambda)^{-n} T_n \left(\sum_{\nu=0}^n (-1)^\nu \mathfrak{N}(\mathfrak{p})^{\frac{(\nu-1)\nu}{2}} \lambda^{n-\nu} T_\nu m \right)^\vee = 0,$$

proving the claim. \square

Assume that \mathcal{M} is an $\mathcal{H}_I \times \mathcal{H}_{I'}$ -module, where

$$\mathcal{H}_{I'} = \mathcal{H}_{\mathbf{Q}}(I_{n-1}^{(m)}, B_{n-1}(k_{\mathfrak{p}})I_{n-1}^{(m)}).$$

Define the inclusions

$$i : \mathcal{H}_I \rightarrow \mathcal{H}_I \otimes \mathcal{H}_{I'}, \quad T \mapsto T \otimes 1$$

and

$$i' : \mathcal{H}_{I'} \rightarrow \mathcal{H}_I \otimes \mathcal{H}_{I'}, \quad T \mapsto 1 \otimes T.$$

Similarly we define the contragredient module \mathcal{M}^\vee by twisting with $\iota \otimes \iota$ and define the map $\cdot^\vee : \mathcal{M} \rightarrow \mathcal{M}^\vee$ as before.

Let $m \in M$ have Hecke roots $\lambda_1, \dots, \lambda_{n-1} \in E$ for \mathcal{H}_I i.e.

$$i(H_{\mathfrak{p}}(\lambda_\nu)) \cdot m = 0$$

for $1 \leq \nu \leq n-1$. Similarly let m have Hecke roots $\lambda'_1, \dots, \lambda'_{n-1} \in E$ for \mathcal{H}_I . We set

$$\underline{\lambda} := (\lambda_1, \dots, \lambda_{n-1})$$

and

$$\underline{\lambda}' := (\lambda'_1, \dots, \lambda'_{n-1})$$

as before.

Then we say that $(m, \underline{\lambda}, \underline{\lambda}', \underline{\mu}, \underline{\nu})$ is of *finite slope* if $i(T_n)$ acts on m via a non-zero scalar, $i'(T_{n-1})$ acts on m via the scalar (16) and if furthermore

$$\nu_{\mathfrak{p}}(\kappa_{\underline{\lambda}} \cdot \kappa_{\underline{\lambda}'}) \in \mathbf{Z} \cup \{\infty\},$$

is finite. If the slope is 0, we call the datum *ordinary*. We remark that in the finite slope case we find a unique $\lambda_n \in E^\times$ such that $i(T_n)$ acts via the scalar

$$\eta_n = \mathfrak{N}(\mathfrak{p})^{-\frac{n(n-1)}{2}} \prod_{\nu=1}^n \lambda_\nu \in E^\times,$$

For a datum of finite slope we set

$$\underline{\lambda}^\vee := (\lambda_n^\vee, \dots, \lambda_2^\vee)$$

and

$$\underline{\lambda}'^\vee := (\lambda_{n-1}'^\vee, \dots, \lambda_1'^\vee)$$

in the notation of Proposition 5.1.

Note that we have in \mathcal{H}_I the identity

$$T_n = \mathfrak{N}(\mathfrak{p})^{-\frac{(n-1)n}{2}} \prod_{i=1}^n U_i = V_{\mathfrak{p},n}$$

by Gritsenko's factorization of $H_{\mathfrak{p}}$. For the operators $V_{\mathfrak{p},\nu}$ and $V_{\mathfrak{p}}$ we have by Lemma 1.2

$$V_{\mathfrak{p},\nu} m^\vee = V_{\mathfrak{p},n} (V_{\mathfrak{p},n-\nu} m)^\vee. \quad (18)$$

for $0 \leq \nu \leq n$ and

$$V_{\mathfrak{p}} m^\vee = V_{\mathfrak{p},n}^{n-1} (V_{\mathfrak{p}} m)^\vee. \quad (19)$$

Hence again if $V_{\mathfrak{p}}$ acts as via multiplication by a unit $\eta \in E^\times$ on m , then $V_{\mathfrak{p}}$ acts on m^\vee via a unit $\eta^\vee \in E^\times$ if and only if $V_{\mathfrak{p},n} = T_n$ acts via a unit $c \in E^\times$ on m .

In the finite slope case the hypotheses of Proposition 1.3 are fulfilled, and we have the dual relation

$$H_{\mathfrak{p}}(\lambda_\nu^\vee) m^\vee = 0$$

for $1 \leq \nu \leq n-1$. Under this condition we have the two modified vectors $m_{\underline{\lambda}}$, resp. $(m^\vee)_{\underline{\lambda}^\vee}$, both eigen vectors of the operator $V_{\mathfrak{p}}$ with the respective eigen values

$$\eta := \mathfrak{N}(\mathfrak{p})^{-\frac{n(n-1)(n-2)}{6}} \cdot \prod_{\nu=1}^{n-1} \lambda_\nu^{n-\nu},$$

and

$$\eta^\vee := \mathfrak{N}(\mathfrak{p})^{-\frac{n(n-1)(n-2)}{6}} \cdot \prod_{\nu=1}^{n-1} (\lambda_{n+1-\nu}^\vee)^{n-\nu}.$$

A direct calculation shows that $V_{\mathfrak{p}}$ acts on

$$(m_{\underline{\Delta}})^\vee \in \mathcal{M}^\vee$$

via η^\vee . Similarly $V_{\mathfrak{p},n}$ acts on $m_{\underline{\Delta}^\vee}^\vee$ via

$$\eta_n^\vee = \mathfrak{N}(\mathfrak{p})^{-\frac{n(n-1)}{2}} \prod_{\nu=1}^n \lambda_\nu^\vee = \mathfrak{N}(\mathfrak{p})^{\frac{n(n-1)}{2}} \prod_{\nu=1}^n \lambda_\nu^{-1} = \eta_n^{-1}.$$

We have

Proposition 5.2. *Let \mathcal{M} be an $\mathcal{H}_I \times \mathcal{H}_{I'}$ -module and let $m \in \mathcal{M}$ be a vector with Hecke roots $\lambda_1, \dots, \lambda_{n-1}$ for $i(\mathcal{H}_I)$ and with Hecke roots $\lambda'_1, \dots, \lambda'_{n-1}$ for $i'(\mathcal{H}_{I'})$. If $(m, \underline{\Delta}, \underline{\Delta}', \underline{\mu}, \underline{\nu})$ is of finite slope, then so is $(m, \underline{\Delta}^\vee, \underline{\Delta}'^\vee, \underline{\check{\mu}}, \underline{\check{\nu}})$ and while $U_{\mathfrak{p}}$ acts on the modified vector*

$$\tilde{m} := \Pi_{\underline{\Delta}}^0 \otimes \Pi_{\underline{\Delta}'^\vee}^0(m)$$

via the scalar $\kappa_{\underline{\Delta}} \cdot \kappa_{\underline{\Delta}'}$, it acts on $(\tilde{m})^\vee$ via the scalar

$$\kappa_{\underline{\Delta}^\vee} \cdot \kappa_{\underline{\Delta}'^\vee}.$$

Furthermore there exists an explicit non-zero constant $C \in E^\times$ with

$$C \cdot (\tilde{m})^\vee = \Pi_{\underline{\Delta}^\vee}^0 \otimes \Pi_{(\underline{\Delta}'^\vee)'}^0(m^\vee).$$

Proof. The relation of the eigen values for the operator $U_{\mathfrak{p}}$ is an immediate consequence of our previous discussion. It remains to show that \cdot^\vee commutes with the modification operator up to a constant. Applying formula (18) to the projection formula yields

$$\begin{aligned} \left(\Pi_{\underline{\Delta}}^0(m) \right)^\vee &= \prod_{i=1}^{n-1} \prod_{\substack{j=1 \\ j \neq i}}^n \eta_n \cdot (\lambda_i \cdot \mathfrak{N}(\mathfrak{p})^{1-j} \cdot V_{\mathfrak{p},n+1-j} + V_{\mathfrak{p},n-j}) \cdot m^\vee = \\ &= \prod_{i=1}^{n-1} \prod_{\substack{j=1 \\ j \neq i}}^n \eta_n \cdot ((\lambda_{n+1-i}^\vee)^{-1} \cdot \mathfrak{N}(\mathfrak{p})^{n+1-j-1} \cdot V_{\mathfrak{p},n+1-j} + V_{\mathfrak{p},n-j}) \cdot m^\vee. \end{aligned}$$

Replacing i with $n+1-i$ and j with $n+1-j$ gives

$$\prod_{i=1}^{n-1} \prod_{\substack{j=1 \\ j \neq i}}^n \eta_n \cdot (\lambda_i^\vee)^{-1} (\lambda_i^\vee \cdot \mathfrak{N}(\mathfrak{p})^{1-j} \cdot V_{\mathfrak{p},j-1} + V_{\mathfrak{p},j}) \cdot m^\vee,$$

proving the claim. \square

5.2 Contragredient cohomology

Now we return to the global situation, i.e. K, K' denote compact open subgroups of the finite adelic groups as before, of levels $I_n^{(m)}$ resp. $I_{n-1}^{(m)}$ at \mathfrak{p} . In this section the long Weyl element w_n is always considered as embedded into the \mathfrak{p} component, i.e.

$$w_n = w_n \otimes_{v\mathfrak{p}} \mathbf{1}_n \in \mathrm{GL}_n(\mathbf{A}_k).$$

Then we define $\iota : \mathrm{GL}_n(\mathbf{A}_k) \rightarrow \mathrm{GL}_n(\mathbf{A}_k)$ as

$$g \mapsto w_n \cdot g^{-t} \cdot w_n,$$

again with w_n only at the \mathfrak{p} -component. We set $K^\vee := \iota(K)$. Note that ι stabilizes the standard maximal compact subgroups as well as their connected components of the identity. Furthermore it stabilizes the center of $G_n(\mathbf{R})$, as well as its connected identity component. As ι is an idempotent, it also fixes Haar measures.

Note that if M_μ is a representation of GL_n , then M_μ^\vee is isomorphic to the contragredient representation $M_{\bar{\mu}}$. We fix such an isomorphism once and for all.

We have a diffeomorphism

$$\iota_K : \mathcal{X}_n(K^\vee) \rightarrow \mathcal{X}_n(K),$$

$$G_n(\mathbf{Q})xK^\vee \mapsto G_n(\mathbf{Q})x^{-t}w_nK.$$

It induces a pullback map on sheaves and we have a natural isomorphism

$$\iota_K^\vee : \iota_K^* \underline{M}_\mu(E) \rightarrow \underline{M}_\mu(E)^\vee,$$

of sheaves on $\mathcal{X}_n(K^\vee)$, which is given on the sections by

$$f \mapsto f^\vee.$$

This morphism induces an isomorphism

$$\iota_K^{\vee*} := \iota_K^\vee \circ \iota_K^* : H_*^q(\mathcal{X}_n(K); \underline{M}_\mu(E)) \rightarrow H_*^q(\mathcal{X}_n(K^\vee); \underline{M}_\mu(E)^\vee),$$

which twists the Hecke action at \mathfrak{p} as in the previous section, i.e. we might identify the right hand side with

$$H_*^q(\mathcal{X}_n(K); \underline{M}_\mu(E))^\vee$$

as $\mathcal{H}_{I(m)}$ -module. This is canonical, if we insist that $\iota_K^{\vee*}$ coincide with the map $\alpha \mapsto \alpha^\vee$. We have the fundamental property that for any $h \in G_n(\mathbf{A}_\mathbf{Q})$

$$iT\iota_h^*(\iota_K^{\vee*} \alpha) = \iota_{\iota(h)K\iota(h)^{-1}}^{\vee*} (iT\iota_{\iota(h)}^* \alpha). \quad (20)$$

We define the matrix

$$\tilde{w} := j(w_{n-1}) \cdot w_n,$$

which again lives only at \mathfrak{p} .

Now observe that translation by \tilde{w} (at \mathfrak{p}) defines a diffeomorphism

$$\begin{aligned} t_{\tilde{w}} : \mathcal{X}_n(j(w_{n-1})Kj(w_{n-1})) &\rightarrow \mathcal{X}_n(K^\vee), \\ G_n(\mathbf{Q})xj(w_{n-1})Kj(w_{n-1}) &\mapsto G_n(\mathbf{Q})x\tilde{w}K^\vee. \end{aligned}$$

Therefore we get

$$\iota_{K'} \circ j = j \circ \iota_K \circ t_{\tilde{w}} \quad (21)$$

In particular the following the square

$$\begin{array}{ccc} H_*^q(\mathcal{X}_n(K); \underline{M}_\mu(E)) & \xrightarrow{\iota_K^{\vee*}} & H_*^q(\mathcal{X}_n(K^\vee); \underline{M}_\mu(E)^\vee) \\ j^* \downarrow & & \uparrow j^* \circ iT t_{\tilde{w}-1}^* \\ H_*^q(\mathcal{X}_{n-1}(K'); j^* \underline{M}_\mu(E)) & \xrightarrow{\iota_{K'}^{\vee*}} & H_*^q(\mathcal{X}_{n-1}(K'^\vee); (j^* \underline{M}_\mu(E)^\vee) \end{array}$$

is commutative.

5.3 Contragredient matrix relation

Finally to deduce the functional equation we need to establish some matrix relations. Consider the matrices

$$n := \begin{pmatrix} -1 & 0 & \cdots & \cdots & 0 \\ -f & 1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & \cdots & 0 & -f & -1 \end{pmatrix} \in \mathrm{GL}_n(k_{\mathfrak{p}})$$

and

$$n' := d_{(x)}^{-1} \cdot \begin{pmatrix} 1 & f & f^2 & \cdots & f^{n-2} \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & f^2 \\ \vdots & & \ddots & \ddots & f \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \cdot d_{(x)} \in \mathrm{GL}_{n-1}(k_{\mathfrak{p}}).$$

Finally set

$$d = \mathrm{diag}(-x_{\mathfrak{p}}, -1, \dots, -1, (-1)^n x_{\mathfrak{p}}^{-1}) \in \mathrm{GL}_{n-1}(k_{\mathfrak{p}}).$$

Then we have, under the usual assumption $\nu_{\mathfrak{p}}(f) \geq m$,

Lemma 5.3. *For any $x_{\mathfrak{p}} \in k_{\mathfrak{p}}$ we have*

$$j(w_{n-1}dn') \cdot \left(d_{(x_{\mathfrak{p}})} h^{(f)} \right)^{-t} w_n \cdot n = j(f^n \cdot \mathbf{1}_{n-1}) f^{1-n} \cdot d_{((-1)^{n-1} x_{\mathfrak{p}}^{-1})} h^{(f)}, \quad (22)$$

with $w_{n-1}dn'w_{n-1} \in I_{n-1}^{(m)}$ and $\det(j(d)n') = 1$, and $n \in I_n^{(m)}$.

We omit the proof, essentially an evaluation of a matrix identity.

5.4 Proof of the functional equation

The map

$$\cdot^\vee : \mathbf{A}_k^{(\infty)\times} \rightarrow \mathbf{A}_k^{(\infty)\times}$$

given by

$$x \mapsto x^\vee := (-1)^{n-1} x^{-1},$$

where the $(-1)^{n-1}$ occurs only in the \mathfrak{p} -component, induces an involution

$$\cdot^\vee : C(K(\mathfrak{p}^\infty)) \rightarrow C(K(\mathfrak{p}^\infty)),$$

and also an involution \cdot^\vee on $C(\mathfrak{p}^\infty)$, which commutes with the covering map (10).

Theorem 5.4. *Let*

$$\lambda \in H_c^{rb_n}(\mathcal{X}_n^{\text{ad}}(K); \underline{M}_\mu(E)) \otimes_E H_c^{rb_{n-1}}(\mathcal{X}_{n-1}^{\text{ad}}(K'); \underline{M}_\nu(E))$$

be an finite slope eigen class for the modified Hecke operator $U_{\mathfrak{p}}$ with eigen value $\kappa_\lambda \in E$. Then we have the functional equation

$$(\mu_\lambda(x))^\vee = \mu_{\lambda^\vee}(x^\vee).$$

By composing the functional equation with the projection τ_ν we get the explicit identity

$$(\tau_\nu(\mu_\lambda(x)))^\vee = \tau_{-\nu}(\mu_{\lambda^\vee}(x^\vee)). \quad (23)$$

Furthermore this functional equation is compatible with the complex functional equation, as the involution we defined is compatible with the notion of automorphic contragredient representations, and also preserves cohomological vectors, i.e. our involution is in particular compatible with our construction of cohomology classes, up to the explicit constant C in Proposition 5.2.

Proof. Fix $x \in \mathbf{A}_k^{(\infty)\times}$ and any nontrivial \mathfrak{p} -power \mathfrak{f} . Write

$$\zeta_\lambda, \zeta_{\lambda'} \in E^\times$$

for the eigen values of $T_n^{\nu_{\mathfrak{p}}(\mathfrak{f})} \otimes 1$ resp. $1 \otimes T_{n-1}^{\nu_{\mathfrak{p}}(\mathfrak{f})}$. Then by the Hecke relation (19) we get an identity

$$\kappa_\lambda(\mathfrak{f}) = \zeta_\lambda^{1-n} \cdot \zeta_{\lambda'}^{-n} \cdot \kappa_{\lambda^\vee}(\mathfrak{f}). \quad (24)$$

Now we have by the definitions, using the relation (20) once,

$$\begin{aligned} \mu_{\lambda^\vee}(x + \mathfrak{f}) &= \kappa_{\lambda^\vee}(\mathfrak{f}) \cdot (iTt_{d(x)\mathfrak{f}t(f)}^*)(s_1^*(iT_{h(f)}) \otimes 1) \mathcal{B}_{h(f),x}^{K^\vee, K'^\vee}(\iota_K^{\vee*} \otimes \iota_{K'}^{\vee*} \lambda) = \\ &= \sum_{\mathcal{X}_{n-1}^\varepsilon(d(x)t(f)K^\vee(\mathfrak{f})t(f)^{-1}d(x)^{-1})[1]} \kappa_{\lambda^\vee}(\mathfrak{f}) \cdot \int (iTt_{d(x)\mathfrak{f}t(f)}^*)(s_1^*(\iota_{\iota(h(f))}^{\vee*})_{K\iota(h(f))^{-1}}(iTt_{\iota(h(f))}\alpha_\varepsilon)) \cup \iota_{K'}^{\vee*}(\text{ad}^* \beta_\varepsilon), \end{aligned}$$

where $K^\vee(\mathfrak{f})$ is defined mutatis mutandis like $K(\mathfrak{f})$, using K^\vee and K'^\vee instead of K and K' . As the embeddings j and s_1 commute with the operator $\iota^{\vee*}$ up to translation by \tilde{w} (cf. (21)), we deduce, again by (20), that for each ε the above integrand equals

$$\iota_{d_{(x)}ft_{(f)}\tilde{w}(\iota(h^{(f)}))K\iota(h^{(f)})^{-1}\tilde{w}^{-1})\cap K'}(d_{(x)}ft_{(f)})^{-1}(iTt_{\iota(d_{(x)}ft_{(f)})}^*)(s_1^*(iTt_{\tilde{w}\iota(h^{(f)})}^*)\alpha_\varepsilon)\cup \text{ad}^*\beta_\varepsilon).$$

We have

$$\iota(d_{(x)}t_{(f)}K^\vee(\mathfrak{f})t_{(f)}^{-1}d_{(x)}^{-1}) = \iota(d_{(x)}t_{(f)})(j^{-1}(\tilde{w}\iota(h^{(f)}))K\iota(h^{(f)})^{-1}\tilde{w}^{-1})\cap K'\iota(t_{(f)}^{-1}d_{(x)}^{-1}),$$

and as ι fixes Haar measures, we conclude that

$$\begin{aligned} \mu_{\lambda^\vee}(x + \mathfrak{f}) &= \iota_{\iota(t_{(f)}d_{(x)})(j^{-1}(\tilde{w}\iota(h^{(f)}))K\iota(h^{(f)})^{-1}\tilde{w}^{-1})\cap K'}(d_{(x)}^{-1}t_{(f)}^{-1}) \\ &\sum \kappa_{\lambda^\vee}(\mathfrak{f}) \cdot \int_{\mathcal{X}_{n-1}(\iota(t_{(f)}d_{(x)})(j^{-1}(\tilde{w}\iota(h^{(f)}))K\iota(h^{(f)})^{-1}\tilde{w}^{-1})\cap K')\iota(d_{(x)}^{-1}t_{(f)}^{-1})[1]} (iTt_{\iota(t_{(f)}fd_{(x)})}^*)(s_1^*(iTt_{\tilde{w}\iota(h^{(f)})}^*)\alpha_\varepsilon) \cup \text{ad}^*\beta_\varepsilon). \end{aligned}$$

We have the matrix relation

$$\iota(t_{(f)}f) \cdot f^n = w_{n-1}f^{-1}t_{(f)}^{-1}w_{n-1} \cdot f^n = t_{(f)}f. \quad (25)$$

Setting in the notation of Lemma 5.3

$$k' := w_{n-1}j(dn')w_{n-1} \in I_{n-1}^{(m)},$$

$$k := n \in I_n^{(m)},$$

the identity (22) reads

$$j(k') \cdot j(\iota(d_{(x)}))\tilde{w}\iota(h^{(f)}) \cdot k = j(f^n \cdot \mathbf{1}_{n-1}) \cdot d_{((-1)^{n-1}x^{-1})}h^{(f)} \cdot f^{1-n}.$$

Therefore

$$\alpha_\varepsilon = iTt_k^*\alpha_\varepsilon,$$

and similarly for β_ε and k'^{-1} , we deduce that

$$\begin{aligned} \mu_{\lambda^\vee}(x + \mathfrak{f}) &= \iota_{t_{(f)}d_{(x^\vee)}(j^{-1}(h^{(f)}K(h^{(f)})^{-1})\cap K')d_{(x^\vee)}^{-1}t_{(f)}^{-1}} \\ &\sum_{\varepsilon} \kappa_{\lambda^\vee}(\mathfrak{f}) \cdot \int_{\mathcal{X}_{n-1}(t_{(f)}d_{(x^\vee)}(j^{-1}(h^{(f)}K(h^{(f)})^{-1})\cap K')d_{(x^\vee)}^{-1}t_{(f)}^{-1})[1]} (iTt_{\iota(t_{(f)}f)f^n d_{(x^\vee)}}^*)(s_1^*(iTt_{h^{(f)}f^{1-n}}^*)\alpha_\varepsilon) \cup \text{ad}^* iTt_{f^{-n}}^*\beta_\varepsilon). \end{aligned}$$

By the matrix relation (25) and the definition of the Hecke operators $T_n \otimes 1$ and $1 \otimes T_{n-1}$, relation (24) shows the claim. \square

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